

# A SUPERCHARACTER THEORY FOR INVOLUTIVE ALGEBRA GROUPS

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**ABSTRACT.** If  $\mathcal{J}$  is a finite-dimensional nilpotent algebra over a finite field  $\mathbb{k}$ , the algebra group  $P = 1 + \mathcal{J}$  admits a (standard) supercharacter theory as defined in [16]. If  $\mathcal{J}$  is endowed with an involution  $\sigma$ , then  $\sigma$  naturally defines a group automorphism of  $P = 1 + \mathcal{J}$ , and we may consider the fixed point subgroup  $C_P(\sigma) = \{x \in P : \sigma(x) = x^{-1}\}$ . Assuming that  $\mathbb{k}$  has odd characteristic  $p$ , we use the standard supercharacter theory for  $P$  to construct a supercharacter theory for  $C_P(\sigma)$ . In particular, we obtain a supercharacter theory for the Sylow  $p$ -subgroups of the finite classical groups of Lie type, and thus extend in a uniform way the construction given by André and Neto in [7, 8] for the special case of the symplectic and orthogonal groups.

## 1. INTRODUCTION

The notion of a supercharacter theory of a finite group was introduced by P. Diaconis and I.M. Isaacs in [16] to generalise the *basic characters* defined by C. André in [2, 3, 4], and the *transition characters* defined by N. Yan in his PhD thesis [23] (see also [24]). Both basic and transition characters were introduced with the aim of approaching the usual character theory of the finite group  $\mathrm{UT}_n(\mathbb{k})$  consisting of  $n \times n$  unimodular upper-triangular matrices over a finite field  $\mathbb{k}$  of characteristic  $p$ . (By “unimodular”, we mean that all diagonal entries are equal to 1; we will refer to  $\mathrm{UT}_n(\mathbb{k})$  simply as a (finite) *unitriangular group*.) The basic idea is to coarsen the usual character theory of a group by replacing irreducible characters with linear combinations of irreducible characters that are constant on a set of clumped conjugacy classes.

Let  $G$  be a finite group, and write  $\mathrm{Irr}(G)$  to denote the set of irreducible characters of  $G$ . (Throughout the paper, all characters are taken over the field  $\mathbb{C}$  of complex numbers.) Let  $\mathcal{K}$  be a partition of  $G$ , and let  $\mathcal{X}$  be a partition of  $\mathrm{Irr}(G)$ . (Here, and throughout this paper, when we use the word “partition”, we require that the parts are all non-empty.) For each  $X \in \mathcal{X}$ , we define

$$(1a) \quad \sigma_X = \sum_{\psi \in X} \psi(1)\psi,$$

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*Date:* February 6, 2015.

2010 *Mathematics Subject Classification.* 20C15, 20D15, 20G40.

This research was made within the activities of the Centro de Estruturas Lineares e Combinatórias and was partially supported by the Fundação para a Ciência e Tecnologia (Portugal).

and note that  $\sum_{X \in \mathcal{X}} \sigma_X = \rho_G$ , the regular character of  $G$ . (Recall that  $\rho_G(g) = 0$  for all  $g \in G$ ,  $g \neq 1$ , and  $\rho_G(1) = |G|$ .) We recall from [16] that the pair  $(\mathcal{X}, \mathcal{K})$  is called a *supercharacter theory* for  $G$  provided that the following conditions hold.

- (S1)  $|\mathcal{X}| = |\mathcal{K}|$ .
- (S2)  $\{1\} \in \mathcal{K}$ .
- (S3) For each  $X \in \mathcal{X}$ , the character  $\sigma_X$  is constant on each member of  $\mathcal{K}$ .

As shown in [16, Lemma 2.1] this definition is equivalent to the following (see [9]). A *supercharacter theory* for a finite group  $G$  is a pair  $(\mathcal{X}, \mathcal{K})$  where  $\mathcal{K}$  is a partition of  $G$ ,  $\mathcal{X}$  is a collection of characters of  $G$ , and the following conditions hold.

- (S1')  $|\mathcal{X}| = |\mathcal{K}|$ .
- (S2') Every irreducible character of  $G$  is a constituent of a unique  $\chi \in \mathcal{X}$ .
- (S3') Every  $\chi \in \mathcal{X}$  is constant on each member of  $\mathcal{K}$ .

We refer to the elements of  $\mathcal{X}$  as the *supercharacters* of  $G$ , and to each  $K \in \mathcal{K}$  as a *superclass* of  $G$ . Regardless of which definition one chooses to work with, it is straightforward to verify that each superclass is a union of conjugacy classes of  $G$  and that each of the partitions  $\mathcal{K}$  and  $\mathcal{X}$  determines the other. The only significant difference between these two definitions is that the second approach can yield supercharacters which are multiples of the characters  $\sigma_X$  defined above.

In the literature to date, one of the main uses of supercharacter theory has been to perform computations when a complete character theory is difficult or impossible to determine. For instance, an explicit computation of the irreducible characters and the conjugacy classes of the finite unitriangular groups  $UT_n(\mathbb{k})$  is known to be a “wild” problem, but André [2] and Yan [23] have developed an applicable supercharacter theory in this situation. (André’s original approach works only when the characteristic of  $\mathbb{k}$  is large enough, although he extends this to the general case in the later paper [4]; Yan’s construction is slightly different and much more elementary, and it yields the same supercharacter theory as André’s.) In [16], Diaconis and Isaacs generalise Yan’s approach in order to extend the supercharacter theory of  $UT_n(\mathbb{k})$  to a much larger class of  $p$ -groups introduced by Isaacs in [21], namely algebra groups over a finite field  $\mathbb{k}$  of characteristic  $p$ . Let  $\mathcal{A}$  be a finite-dimensional associative  $\mathbb{k}$ -algebra (with identity), and write  $\mathcal{A}^\times$  to denote the unit group of  $\mathcal{A}$  (that is, the group of invertible elements of  $\mathcal{A}$ ). Following the terminology of [21], given any nilpotent subalgebra  $\mathcal{J}$  of  $\mathcal{A}$ , the *algebra group* based on  $\mathcal{J}$  is the multiplicative subgroup  $1 + \mathcal{J}$  of  $\mathcal{A}^\times$ ; notice that a subalgebra of  $\mathcal{A}$  is not required to contain the identity (it is simply a multiplicatively closed vector subspace of  $\mathcal{A}$ ). We note that  $\mathbb{k} \cdot 1 + \mathcal{J}$  is a (local) subalgebra of  $\mathcal{A}$ , and that  $P = 1 + \mathcal{J}$  is a (normal) Sylow  $p$ -subgroup of the unit group  $(\mathbb{k} \cdot 1 + \mathcal{J})^\times$ ; indeed,  $(\mathbb{k} \cdot 1 + \mathcal{J})^\times$  is isomorphic to the direct product  $\mathbb{k}^\times \times P$ . In fact, it is shown in [5, Theorem 1.5] that a finite group is an algebra group over  $\mathbb{k}$  if

and only if it is a Sylow  $p$ -subgroup of the unit group of some finite-dimensional  $\mathbb{k}$ -algebra  $\mathcal{A}$ . These algebra groups generalise the finite unitriangular groups over  $\mathbb{k}$ ; in this standard example, we let  $\mathcal{A} = \mathcal{M}_n(\mathbb{k})$  be the  $\mathbb{k}$ -algebra consisting of all  $n \times n$  matrices with entries in  $\mathbb{k}$ , so that  $\mathcal{A}^\times = \mathrm{GL}_n(\mathbb{k})$  is the general linear group consisting of all invertible matrices in  $\mathcal{M}_n(\mathbb{k})$ . Then,  $\mathrm{UT}_n(\mathbb{k}) = 1 + \mathcal{J}$  is the algebra group based on the nilpotent subalgebra  $\mathcal{J} = \mathfrak{ut}_n(\mathbb{k})$  of  $\mathcal{M}_n(\mathbb{k})$  which consists of all strictly upper-triangular matrices.

The primary aim of this paper is to develop a supercharacter theory for another family of  $p$ -groups which are associated with finite-dimensional nilpotent  $\mathbb{k}$ -algebras with involution. These  $p$ -groups include the Sylow  $p$ -subgroups of the finite classical groups of Lie type, and our construction is motivated by the methods used by C. André and A.M. Neto in [7, 8, 9] for the particular case of the Sylow  $p$ -subgroups of the symplectic group  $Sp_{2m}(\mathbb{k})$ , and the orthogonal groups  $O_{2m}^+(\mathbb{k})$  and  $O_{2m+1}(\mathbb{k})$  (see below). We assume that  $\mathbb{k}$  is a finite field of odd characteristic  $p$ , and let  $\mathcal{A}$  is a finite-dimensional  $\mathbb{k}$ -algebra endowed with an involution. We recall that an *involution* on  $\mathcal{A}$  is a map  $\sigma: \mathcal{A} \rightarrow \mathcal{A}$  satisfying the following conditions:

- (1)  $\sigma(a + b) = \sigma(a) + \sigma(b)$  for all  $a, b \in \mathcal{A}$ ;
- (2)  $\sigma(ab) = \sigma(b)\sigma(a)$  for all  $a, b \in \mathcal{A}$ ;
- (3)  $\sigma^2(a) = a$  for all  $a \in \mathcal{A}$ .

We note that an involution  $\sigma$  is not required to be  $\mathbb{k}$ -linear; however, we will assume that the field  $\mathbb{k} = \mathbb{k} \cdot 1$  is preserved by  $\sigma$ <sup>1</sup>. Then,  $\sigma$  defines a field automorphism of  $\mathbb{k}$  which is either the identity or has order 2; we say that  $\sigma$  is *of the first kind* if  $\sigma$  fixes  $\mathbb{k}$ , and *of the second kind* if its restriction  $\sigma|_{\mathbb{k}}$  to  $\mathbb{k}$  has order 2. In any case, we let  $\mathbb{k}^\sigma = \{\alpha \in \mathbb{k}: \sigma(\alpha) = \alpha\}$  denote the  $\sigma$ -fixed subfield of  $\mathbb{k}$ , and consider that  $\mathcal{A}$  is a finite dimensional associative  $\mathbb{k}^\sigma$ -algebra. We observe that  $\sigma$  is of the second kind if and only if the field extension  $\mathbb{k}^\sigma \subseteq \mathbb{k}$  has degree 2, and  $\sigma: \mathbb{k} \rightarrow \mathbb{k}$  is the *Frobenius map* defined by the mapping  $\alpha \mapsto \alpha^q$  where  $q = |\mathbb{k}^\sigma|$ ; for simplicity of writing, we will use the bar notation  $\bar{\alpha} = \alpha^q$  for  $\alpha \in \mathbb{k}$ .

An important example occurs in the case where  $\mathcal{A} = \mathcal{M}_n(F)$  is endowed with the canonical *transpose involution* given by the mapping  $a \mapsto a^T$  where  $a^T$  denotes the transpose of  $a \in \mathcal{M}_n(F)$ . More generally, let  $q = |\mathbb{k}^\sigma|$ , let  $\mathrm{Fr}_q: \mathcal{M}_n(\mathbb{k}) \rightarrow \mathcal{M}_n(\mathbb{k})$  be the Frobenius morphism defined by  $\mathrm{Fr}_q(a_{ij}) = (\bar{a}_{ij}) = (a_{ij}^q)$  for all  $(a_{ij}) \in \mathcal{M}_n(\mathbb{k})$ , and set  $a^* = \mathrm{Fr}_q(a)^T$  for all  $a \in \mathcal{M}_n(\mathbb{k})$ . Then, the mapping  $a \mapsto a^*$  defines an involution on  $\mathcal{M}_n(\mathbb{k})$ ; notice that, if  $\mathbb{k}^\sigma = \mathbb{k}$ , then  $a^* = a^T$  for all  $a \in \mathcal{M}_n(\mathbb{k})$ . If  $\sigma: \mathcal{M}_n(\mathbb{k}) \rightarrow \mathcal{M}_n(\mathbb{k})$  is an involution of the first kind, then there exists  $u \in \mathrm{GL}_n(\mathbb{k})$  with  $u^T = \pm u$  and such that  $\sigma(a) = u^{-1}a^Tu$  for all  $a \in \mathcal{M}_n(\mathbb{k})$ ; moreover, the matrix  $u$  is uniquely determined up to a factor in  $\mathbb{k}^\times$ . On the other hand, if  $\sigma: \mathcal{M}_n(\mathbb{k}) \rightarrow \mathcal{M}_n(\mathbb{k})$

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<sup>1</sup>This essential assumption is missing in the definition given in [5]; however, it is implicit throughout that paper and all results are valid under this hypothesis. The first author is grateful to I.M. Isaacs for pointing this out to him.

is an involution of the second kind, then there exists  $u \in \mathrm{GL}_n(\mathbb{k})$  with  $u^* = u$  and such that  $\sigma(a) = u^{-1}a^*u$  for all  $a \in \mathcal{M}_n(\mathbb{k})$ ; moreover, the matrix  $u$  is uniquely determined up to a factor in  $(\mathbb{k}^\sigma)^\times$ . [The proofs can be found in the book [22] by M.-A. Knus *et al.* (see, in particular, Propositions 2.19 and 2.20) where the complete classification of involutions is also given for arbitrary central  $\mathbb{k}$ -algebras (see Propositions 2.7 and 2.18).] For simplicity, for  $u \in \mathrm{GL}_n(\mathbb{k})$  as above, we will denote by  $\sigma_u$  the involution on  $\mathcal{M}_n(F)$  given by the mapping  $a \mapsto u^{-1}a^*u$ ; as usual, we say that  $\sigma_u$  is *symplectic* if  $\sigma_u$  is of the first kind and  $u^T = -u$ , *orthogonal* if  $\sigma_u$  is of the first kind and  $u^T = u$ , and *unitary* if  $\sigma_u$  is of the second kind and  $u^* = u$ .

In the general situation, consider the unit group  $\mathcal{A}^\times$  of the  $\mathbb{k}$ -algebra  $\mathcal{A}$ . Then, for any involution  $\sigma: \mathcal{A} \rightarrow \mathcal{A}$ , the cyclic group  $\langle \sigma \rangle$  acts on  $\mathcal{A}^\times$  as a group of automorphisms by means of  $x^\sigma = \sigma(x^{-1})$  for all  $x \in \mathcal{A}^\times$  ( $x^\sigma$  should not be confused with  $\sigma(x)$ ). For any  $\sigma$ -invariant subgroup  $H$  of  $\mathcal{A}^\times$ , we denote by  $C_H(\sigma)$  the subgroup of  $H$  consisting of all  $\sigma$ -fixed elements; that is,  $C_H(\sigma) = \{x \in H: x^\sigma = x\} = \{x \in H: \sigma(x^{-1}) = x\}$ . In the case where  $\mathcal{A} = \mathcal{M}_n(\mathbb{k})$ , an arbitrary involution  $\sigma: \mathcal{M}_n(\mathbb{k}) \rightarrow \mathcal{M}_n(\mathbb{k})$  defines a group  $C_{\mathrm{GL}_n(\mathbb{k})}(\sigma)$  which is isomorphic to one of the *finite classical groups of Lie type* (defined over  $\mathbb{k}$ ): the *symplectic group*  $Sp_{2m}(\mathbb{k})$  if  $\sigma$  is symplectic, the *orthogonal groups*  $O_{2m}^+(\mathbb{k})$ ,  $O_{2m+1}(\mathbb{k})$ , or  $O_{2m+2}^-(\mathbb{k})$  if  $\sigma$  is orthogonal, and the *unitary group*  $U_n(\mathbb{k})$  if  $\sigma$  is unitary. [For the details on the definition of the classical groups, we refer to Chapter I the book [15] by R. Carter.] In fact, up to isomorphism, these groups may be defined by the involution  $\sigma = \sigma_u$  where  $u \in \mathrm{GL}_n(\mathbb{k})$  is the matrix defined as follows; here,  $J_m$  denotes the  $m \times m$  matrix with 1's along the anti-diagonal and 0's elsewhere.

- (1) For  $Sp_{2m}(\mathbb{k})$ , we choose  $u = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$ .
- (2) For  $O_{2m}^+(\mathbb{k})$  or  $O_{2m+1}(\mathbb{k})$ , we choose  $u = J_n$  where, either  $n = 2m$ , or  $n = 2m + 1$ .
- (3) For  $O_{2m+2}^-(\mathbb{k})$ , we choose  $u = \begin{pmatrix} 0 & 0 & J_m \\ 0 & c & 0 \\ J_m & 0 & 0 \end{pmatrix}$  where  $c = \begin{pmatrix} 1 & 0 \\ 0 & -\varepsilon \end{pmatrix}$  for  $\varepsilon \in \mathbb{k}^\times \setminus (\mathbb{k}^\times)^2$ .
- (4) For  $U_n(\mathbb{k})$ , we choose  $u = J_n$ .

We refer to  $\sigma = \sigma_u$  (for this matrix  $u$ ) as a *canonical involution* on  $\mathcal{M}_n(\mathbb{k})$ .

As we mentioned above, our main goal in this paper is to develop a supercharacter theory for the group  $C_P(\sigma)$  in the case where  $P$  is a  $\sigma$ -invariant algebra subgroup of  $\mathcal{A}^\times$ . Our construction is given in terms of the supercharacter theory of  $P$ , and extends the results of [7, 8, 9] in the particular case where  $P = \mathrm{UT}_n(\mathbb{k})$  is the unitriangular group over  $\mathbb{k}$  and  $C_P(\sigma)$  is the Sylow  $p$ -subgroup of  $Sp_{2m}(\mathbb{k})$ ,  $O_{2m}^+(\mathbb{k})$  or  $O_{2m+1}(\mathbb{k})$ . More generally, our construction applies to the particular case where  $\mathcal{A} = \mathcal{M}_n(\mathbb{k})$ , and  $\sigma: \mathcal{M}_n(\mathbb{k}) \rightarrow \mathcal{M}_n(\mathbb{k})$  is any canonical involution. In this situation, it is well-known that the Sylow  $p$ -subgroups of  $C_{\mathrm{GL}_n(\mathbb{k})}(\sigma)$  are conjugate to the  $\sigma$ -fixed subgroup  $C_P(\sigma)$  where  $P$  is, either is the unitriangular subgroup  $\mathrm{UT}_n(\mathbb{k})$  of  $\mathrm{GL}_n(\mathbb{k})$ , or the subgroup of  $\mathrm{UT}_n(\mathbb{k})$  consisting of all unimodular upper-triangular matrices with  $(m+1, m+2)$ th position equal to zero. The former situation occurs only if  $G$  is the orthogonal group  $O_{2m+2}^-(q)$ ; indeed, the unitriangular group is not invariant for the corresponding involution. [In this case,

the supercharacter theory of  $P$  has a slightly different parametrization than that of  $\mathrm{UT}_n(F)$ , and thus the supercharacter theory of  $C_P(\sigma)$  has to be described separately; we leave this description as an exercise for the reader.]

To conclude this introduction, we mention that supercharacter theories have proven to be relevant outside the realm of finite group theory. For instance, as shown in [16] these notions can be used to obtain a more general theory of spherical functions and Gelfand pairs. Another application may be found in [11] where the supercharacter theory of  $\mathrm{UT}_n(\mathbb{k})$  is applied to study random walks on upper-triangular matrices. In a different direction, recent work has revealed deep connections between the supercharacter theory of  $\mathrm{UT}_n(\mathbb{k})$  and the Hopf algebra of symmetric functions of noncommuting variables (see [1, 12, 14]). We hope that analogous applications and connections could be derived using the supercharacter theories developed in this paper (see the recent paper [13] by C. Benedetti). Finally, we also mention the relation between supercharacter theories and Schur rings discovered by O. Hendrickson in [19], and the applications of supercharacter theories of finite abelian groups to exponential sums in number theory (see [17, 18]).

*Basic notation and terminology.* Throughout the paper, we let  $\mathbb{k}$  denote a finite field with odd characteristic  $p$ , let  $\mathcal{A}$  be a finite-dimensional  $\mathbb{k}$ -algebra endowed with an involution  $\sigma: \mathcal{A} \rightarrow \mathcal{A}$ , and let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ . Let  $\mathcal{A}^\times$  denote the unit group of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$  be the algebra subgroup of  $\mathcal{A}^\times$  based on  $\mathcal{J}$ . Then,  $P$  is  $\sigma$ -invariant with respect to the action given by

$$(1b) \quad x^\sigma = \sigma(x^{-1})$$

for all  $x \in \mathcal{A}^\times$ . As usual, we write  $C_P(\sigma)$  to denote the subgroup of  $P$  consisting of all  $\sigma$ -fixed elements, that is,

$$C_P(\sigma) = \{x \in P: x^\sigma = x\}.$$

We define the *Cayley transform*  $\Phi: \mathcal{J} \rightarrow P$  by the rule

$$(1c) \quad \Phi(a) = (1 + a)(1 - a)^{-1} = 1 + 2a(1 - a)^{-1}$$

for all  $a \in \mathcal{J}$ ; notice that  $(1 - a)^{-1} = 1 + a(1 - a)^{-1}$  for all  $a \in \mathcal{J}$ . Since  $p$  is odd, this map is bijective, and its inverse  $\Psi: P \rightarrow \mathcal{J}$  is given by

$$(1d) \quad \Psi(x) = (x - 1)(x + 1)^{-1}$$

for all  $x \in P$ . It is clear that  $\Phi(\sigma(a)) = \sigma(\Phi(a))$  for all  $a \in \mathcal{J}$ , and so the Cayley transform restricts to a bijective map  $\Phi: C_{\mathcal{J}}(\sigma) \rightarrow C_P(\sigma)$  where we set

$$C_{\mathcal{J}}(\sigma) = \{a \in \mathcal{J}: \sigma(a) = -a\};$$

notice that  $C_{\mathcal{J}}(\sigma)$  is a vector space over the  $\sigma$ -fixed subfield  $\mathbb{k}^\sigma$  of  $\mathbb{k}$ . Throughout the paper, we consider the action of  $\sigma$  on  $\mathcal{J}$  defined by

$$(1e) \quad a^\sigma = -\sigma(a)$$

for all  $a \in \mathcal{J}$ , so that  $C_{\mathcal{J}}(\sigma) = \{a \in \mathcal{J}: a^\sigma = a\}$  is the (additive) subgroup of  $\mathcal{J}$  consisting of all  $\sigma$ -fixed elements. We observe that this action commutes with  $\Phi$ , that is,

$$(1f) \quad \Phi(a^\sigma) = \Phi(a)^\sigma$$

for all  $a \in \mathcal{J}$ ; notice also that  $\Psi(x^\sigma) = \Psi(x)^\sigma$  for all  $x \in P$ .

On the other hand, we denote by  $\mathcal{J}^\circ$  the *dual group* of  $\mathcal{J}^+$  which by definition consists of all linear characters  $\lambda: \mathcal{J}^+ \rightarrow \mathbb{C}$  of the additive group  $\mathcal{J}^+$  of  $\mathcal{J}$ ; since  $\mathcal{J}^+$  is an abelian group, it is a standard fact that  $\mathcal{J}^\circ$  is the set  $\text{Irr}(\mathcal{J}^+)$  of all irreducible characters of  $\mathcal{J}^+$ . We note that  $\mathcal{J}^\circ$  is an abelian group with respect to the product of characters defined by  $(\lambda\mu)(a) = \lambda(a)\mu(a)$  for all  $\lambda, \mu \in \mathcal{J}^\circ$  and all  $a \in \mathcal{J}$ ; in particular, notice that  $\lambda^2(a) = \lambda(a)\lambda(a) = \lambda(2a)$  for all  $\lambda \in \mathcal{J}^\circ$  and all  $a \in \mathcal{J}$ . For every  $\lambda \in \mathcal{J}^\circ$ , we define the linear character  $\lambda^\sigma \in \mathcal{J}^\circ$  by

$$(1g) \quad \lambda^\sigma(a) = \lambda(a^\sigma) = \lambda(-\sigma(a))$$

for all  $a \in \mathcal{J}$ . This clearly defines an action of  $\sigma$  on  $\mathcal{J}^\circ$ , and thus we can define the  $\sigma$ -fixed subgroup  $C_{\mathcal{J}^\circ}(\sigma) = \{\lambda \in \mathcal{J}^\circ: \lambda^\sigma = \lambda\}$  of  $\mathcal{J}^\circ$ . However, we prefer to realise this subgroup as the dual group  $C_{\mathcal{J}}(\sigma)^\circ$  of the additive group  $C_{\mathcal{J}}(\sigma)^+$  of  $C_{\mathcal{J}}(\sigma)$ . In fact, it is easily seen that  $\mathcal{J}$  decomposes as the direct sum  $\mathcal{J} = C_{\mathcal{J}}(\sigma) \oplus [\mathcal{J}, \sigma]$  where  $[\mathcal{J}, \sigma] = \{a + \sigma(a): a \in \mathcal{J}\}$ , and thus  $C_{\mathcal{J}}(\sigma)^\circ$  can be naturally identified with the orthogonal subgroup  $[\mathcal{J}, \sigma]^\perp$ ; for any additive subgroup  $\mathcal{I}$  of  $\mathcal{J}$ , the *orthogonal subgroup*  $\mathcal{I}^\perp$  is defined by  $\mathcal{I}^\perp = \{\lambda \in \mathcal{J}^\circ: \mathcal{I} \subseteq \ker(\lambda)\}$ . In light of the above identification, we see that

$$(1h) \quad C_{\mathcal{J}}(\sigma)^\circ = \{\lambda \in \mathcal{J}^\circ: \lambda^\sigma = \lambda\};$$

indeed, the Eq. (1g) implies that for every  $\lambda \in \mathcal{J}^\circ$  we have  $\lambda^\sigma = \lambda$  if and only if  $\lambda(a + \sigma(a)) = 1$  for all  $a \in \mathcal{J}$ .

## 2. SUPERCLASSES

Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . Then, right multiplication defines a right action of  $P$  on  $\mathcal{J}$ , whereas left multiplication defines a left action of  $P$  on  $\mathcal{J}$ ; these two actions are compatible in the sense that  $(xa)y = x(ay)$  for all  $x, y \in P$  and all  $a \in \mathcal{J}$ . It follows that  $\mathcal{J}$  decomposes as a disjoint union of *two-sided orbits*  $PaP$  for  $a \in \mathcal{J}$ . Then, the *superclasses* of the algebra group  $P$  are defined to be the subsets of the form  $1 + PaP$  where  $a \in \mathcal{J}$ ; we write  $\text{SCl}(P)$  to denote the set of all superclasses of  $P$ . We note that, for any  $a \in \mathcal{J}$ , the set  $PaP$  is an orbit for the natural action of  $P \times P$  on  $\mathcal{J}$  given by  $(x, y) \cdot a = xay^{-1}$  for all  $a \in \mathcal{J}$  and all  $x, y \in P$ , and that every superclass is a (disjoint) union of conjugacy classes. In fact,

every two-sided orbit on  $\mathcal{J}$  is a disjoint union of *conjugation orbits* where the *conjugation action*  $P \times \mathcal{J} \rightarrow \mathcal{J}$  is defined by the mapping  $(x, a) \mapsto xax^{-1}$ .

The purpose of this section is to define superclasses of the  $\sigma$ -fixed subgroup  $C_P(\sigma)$  of  $P$ , and the most natural way of defining them is to consider the non-empty intersections  $\widehat{\mathcal{K}} \cap C_P(\sigma)$  where  $\widehat{\mathcal{K}}$  is a superclass of  $P$ . [Throughout the paper, we shall use the hat notation  $\widehat{\mathcal{K}}$  for superclasses of  $P$ , and reserve the notation  $\mathcal{K}$  for the intersection  $\widehat{\mathcal{K}} \cap C_P(\sigma)$ .] Obviously, the intersection  $\widehat{\mathcal{K}} \cap C_P(\sigma)$  is non-empty if and only if  $\widehat{\mathcal{K}}$  contains an element which is fixed by  $\sigma$ , and we shall prove that this is equivalent to requiring that the superclass  $\widehat{\mathcal{K}}$  is  $\sigma$ -invariant. We start by proving an alternative description of the superclasses of  $P$  in terms of the Cayley transform  $\Phi: \mathcal{J} \rightarrow P$ ; this is crucial for our work because  $\Phi$  defines a bijection from  $C_{\mathcal{J}}(\sigma)$  to  $C_P(\sigma)$  (whereas the standard mapping  $a \mapsto 1 + a$  does not).

**Lemma 2.1.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . If  $a \in \mathcal{J}$  and  $\widehat{\mathcal{K}} \in \text{SCl}(P)$  is the superclass which contains  $x = \Phi(a)$ , then  $\widehat{\mathcal{K}} = 1 + P(2a)P = \Phi(PaP)$ . In particular,  $\text{SCl}(P) = \{\Phi(PaP): a \in \mathcal{J}\}$ .*

*Proof.* Since  $x = \Phi(a) = 1 + 2a(1 - a)^{-1}$ , we clearly have  $x \in 1 + P(2a)P$ , and thus  $\widehat{\mathcal{K}} = 1 + P(2a)P$ . If  $y, z \in P$ , then  $\Phi(yaz) \in 1 + P(2yaz)P = 1 + P(2a)P$ , and thus  $\Phi(PaP) \subseteq 1 + P(2a)P$ . The result follows because  $\Phi$  is bijective and  $|PaP| = |P(2a)P|$ .  $\square$

Next, we observe that the cyclic group  $\langle \sigma \rangle$  acts on the set  $\text{SCl}(P)$ .

**Lemma 2.2.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . If  $\widehat{\mathcal{K}} \in \text{SCl}(P)$ , then  $\widehat{\mathcal{K}}^\sigma \in \text{SCl}(P)$ ; in fact, if  $\widehat{\mathcal{K}} = \Phi(PaP)$  for  $a \in \mathcal{J}$ , then  $\widehat{\mathcal{K}}^\sigma = \Phi(Pa^\sigma P)$ .*

*Proof.* It is enough to use Eq. (1f) since  $(xay)^\sigma = y^{-\sigma}a^\sigma x^{-\sigma}$  for all  $x, y \in P$ ; as usual, we write  $z^{-\sigma} = (z^{-1})^\sigma$  for all  $z \in P$ .  $\square$

Henceforth, we denote by  $\text{SCl}_\sigma(P)$  the subset of  $\text{SCl}(P)$  consisting of all  $\sigma$ -invariant superclasses of  $P$ . By [20, Corollary 13.10], every conjugacy class  $\mathcal{C}$  of  $C_P(\sigma)$  is the intersection  $\mathcal{C} = \widehat{\mathcal{C}} \cap C_P(\sigma)$  for some  $\sigma$ -invariant conjugacy class  $\widehat{\mathcal{C}}$  of  $P$ , and moreover the mapping  $\widehat{\mathcal{C}} \mapsto \widehat{\mathcal{C}} \cap C_P(\sigma)$  defines a bijection between the set of  $\sigma$ -invariant conjugacy classes of  $P$  and the set of conjugacy classes of  $C_P(\sigma)$ . Therefore, for every superclass  $\widehat{\mathcal{K}} \in \text{SCl}(P)$ , either the intersection  $\widehat{\mathcal{K}} \cap C_P(\sigma)$  is empty, or it is a union of conjugacy classes of  $C_P(\sigma)$ ; this is one of the conditions which should be satisfied by any set of superclasses. We define a *superclass* of  $C_P(\sigma)$  to be a non-empty intersection  $\widehat{\mathcal{K}} \cap C_P(\sigma)$  for  $\widehat{\mathcal{K}} \in \text{SCl}(P)$ , and denote by  $\text{SCl}(C_P(\sigma))$  the set of all superclasses of  $C_P(\sigma)$ . [Eventually, we will define the supercharacters of  $C_P(\sigma)$ , and we will see that these definitions are compatible with the general definition of a supercharacter theory.] We have the following result.

**Proposition 2.3.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , let  $P = 1 + \mathcal{J}$ , and let  $\widehat{\mathcal{K}} \in \text{SCl}(P)$ . Then, the intersection  $\widehat{\mathcal{K}} \cap C_P(\sigma)$  is non-empty if and only if the superclass  $\widehat{\mathcal{K}}$  is  $\sigma$ -invariant.*

*Proof.* If  $\widehat{\mathcal{K}} \cap C_P(\sigma)$  is non-empty and  $x \in \widehat{\mathcal{K}} \cap C_P(\sigma)$ , then  $x \in \widehat{\mathcal{K}} \cap \widehat{\mathcal{K}}^\sigma$ . Since  $\widehat{\mathcal{K}}^\sigma$  is a superclass of  $P$ , it follows that  $\widehat{\mathcal{K}} = \widehat{\mathcal{K}}^\sigma$ . Conversely, suppose that  $\widehat{\mathcal{K}} = \widehat{\mathcal{K}}^\sigma$ , and let  $a \in \mathcal{J}$  be such that  $\Phi(a) \in \widehat{\mathcal{K}}$ . By the previous lemma, we have  $PaP = Pa^\sigma P$ . Now, we consider the automorphism of the group  $P \times P$  defined by the mapping  $(x, y) \mapsto (x, y)^\sigma = (y^\sigma, x^\sigma)$ , and observe that

$$((x, y) \cdot a)^\sigma = (xay^{-1})^\sigma = y^\sigma a^\sigma x^{-\sigma} = (y^\sigma, x^\sigma) \cdot a^\sigma = (x, y)^\sigma \cdot a^\sigma$$

for all  $x, y \in P$  and all  $a \in \mathcal{J}$ . Thus, since  $P \times P$  acts transitively on  $PaP$  (and since  $2 \nmid |P|$ ), Glauberman's Lemma (see [20, Lemma 13.8]) implies that there exists  $b \in PaP$  such that  $b^\sigma = b$ . By Eq. (1f) and by the previous lemma, we conclude that the element  $x = \Phi(b) \in P$  satisfies  $x^\sigma = x$  and lies in  $\widehat{\mathcal{K}}$ .  $\square$

It follows that

$$(2a) \quad \text{SCl}(C_P(\sigma)) = \{\widehat{\mathcal{K}} \cap C_P(\sigma) : \widehat{\mathcal{K}} \in \text{SCl}_\sigma(P)\};$$

moreover, the mapping  $\widehat{\mathcal{K}} \mapsto \widehat{\mathcal{K}} \cap C_P(\sigma)$  defines a bijection between  $\text{SCl}_\sigma(P)$  and  $\text{SCl}(C_P(\sigma))$ . As we observed above, since every superclass of  $P$  is a union of conjugacy classes, [20, Corollary 13.10] implies that every superclass of  $C_P(\sigma)$  is also a union of conjugacy classes. Indeed, the following result also implies that every superclass of  $C_P(\sigma)$  is invariant under conjugation.

**Theorem 2.4.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . If  $a \in C_{\mathcal{J}}(\sigma)$  and  $\widehat{\mathcal{K}} \in \text{SCl}(P)$  contains  $x = \Phi(a) \in C_P(\sigma)$ , then*

$$\widehat{\mathcal{K}} \cap C_P(\sigma) = \{\Phi(zaz^{-\sigma}) : z \in P\}$$

where we write  $z^{-\sigma} = (z^{-1})^\sigma$  for all  $z \in P$ .

*Proof.* By Lemma 2.1 and Eq. (1f), we see that  $\widehat{\mathcal{K}} \cap C_P(\sigma) = \{\Phi(u) : u \in PaP, u^\sigma = u\}$ . As in the proof of Proposition 2.3, we consider  $\sigma$  as the automorphism of  $P \times P$  given by the mapping  $(y, z) \mapsto (z^\sigma, y^\sigma)$ . It follows by [20, Corollary 13.9] that the set  $\{u \in PaP : u^\sigma = u\}$  is an orbit for the action of the subgroup  $C_{P \times P}(\sigma) = \{(z, z^\sigma) : z \in P\}$ . In other words, we have  $\{u \in PaP : u^\sigma = u\} = \{zuz^{-\sigma} : z \in P\}$ , and the result follows.  $\square$

We note that the algebra group  $P$  acts on the left of  $C_{\mathcal{J}}(\sigma)$  by the rule  $x \cdot a = x^{-1}ax^\sigma$  for all  $x \in P$  and all  $a \in C_{\mathcal{J}}(\sigma)$ . Then, the previous theorem asserts that the superclass of  $C_P(\sigma)$  which contains an element  $x \in C_P(\sigma)$  is the image  $\Phi(\Omega_P(a))$  of the orbit  $\Omega_P(a) = \{x^{-1}ax^\sigma : x \in P\}$  which contains the element  $a \in C_{\mathcal{J}}(\sigma)$  such that  $x = \Phi(a)$ .

### 3. SUPERCHARACTERS

In this section we define the supercharacters of the group  $C_P(\sigma)$  where  $P = 1 + \mathcal{J}$  and  $\mathcal{J}$  is a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ . We start by summarising the construction of the supercharacters of the algebra group  $P$ ; our main reference is [16]. Let  $\mathcal{J}^\circ$  be the dual group of  $\mathcal{J}^+$ , and for every  $\lambda \in \mathcal{J}^\circ$  and every  $x \in P$  define the linear characters  $\lambda x, x\lambda \in \mathcal{J}^\circ$  by the formulas  $(\lambda x)(a) = \lambda(ax^{-1})$  and  $(x\lambda)(a) = \lambda(x^{-1}a)$  for all  $a \in \mathcal{J}$ . These actions are compatible in the sense that  $(x\lambda)y = x(\lambda y)$  for all  $x, y \in P$  and all  $\lambda \in \mathcal{J}^\circ$ , and thus  $\mathcal{J}^\circ$  decomposes as a disjoint union of *two-sided orbits*  $P\lambda P$  for  $\lambda \in \mathcal{J}^\circ$ . Furthermore, every two-sided orbit on  $\mathcal{J}^\circ$  is a disjoint union of *conjugation orbits* where the *conjugation action*  $P \times \mathcal{J}^\circ \rightarrow \mathcal{J}^\circ$  is defined by the mapping  $(x, \lambda) \mapsto x\lambda x^{-1}$ . We also observe that  $P\lambda P$  is an orbit for the natural action of  $P \times P$  on  $\mathcal{J}^\circ$  given by  $(x, y) \cdot \lambda = x\lambda y^{-1}$  for all  $\lambda \in \mathcal{J}^\circ$  and all  $x, y \in P$ .

The supercharacters of  $P$  are in one-to-one correspondence with the two-sided orbits on  $\mathcal{J}^\circ$ . For every  $\lambda \in \mathcal{J}^\circ$ , the *supercharacter*  $\widehat{\chi}_\lambda$  which corresponds to  $P\lambda P$  is given by the formula

$$(3a) \quad \widehat{\chi}_\lambda(x) = \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda P} \mu(x - 1)$$

for all  $x \in P$ ; we set  $SCh(P) = \{\widehat{\chi}_\lambda : \lambda \in \mathcal{J}^\circ\}$ . [As for superclasses, we shall use the hat notation  $\widehat{\chi}$  for characters of  $P$ , and reserve the non-hat notation for the characters of  $C_P(\sigma)$ ; in particular,  $\widehat{\chi}_\lambda$  will always refer to the supercharacter of  $P$  associated with the linear character  $\lambda \in \mathcal{J}^\circ$  of  $\mathcal{J}$ .] It is clear that every supercharacter  $\widehat{\chi} \in SCh(P)$  has a constant value on each superclass of  $P$ , and that for every  $\lambda, \mu \in \mathcal{J}^\circ$  we have  $\langle \widehat{\chi}_\lambda, \widehat{\chi}_\mu \rangle = 0$  unless  $P\lambda P = P\mu P$ , in which case we clearly have  $\widehat{\chi}_\lambda = \widehat{\chi}_\mu$ . [If  $G$  is any finite group, we define the *Frobenius scalar product*

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{x \in G} \alpha(x) \overline{\beta(x)}$$

for all complex-valued functions  $\alpha$  and  $\beta$  defined on  $G$ .] In fact, it is straightforward to check that the regular character  $\varrho_P$  of  $P$  decomposes as the orthogonal sum  $\varrho_P = \sum_{\widehat{\chi} \in SCh(P)} n_{\widehat{\chi}} \widehat{\chi}$  where  $n_{\widehat{\chi}} = \widehat{\chi}(1)/\langle \widehat{\chi}, \widehat{\chi} \rangle$  for all  $\widehat{\chi} \in SCh(P)$ . In fact, for every  $\lambda \in \mathcal{J}^\circ$ , we have

$$\widehat{\chi}_\lambda(1) = |P\lambda| \quad \text{and} \quad \langle \widehat{\chi}_\lambda, \widehat{\chi}_\lambda \rangle = |P\lambda \cap \lambda P|$$

(see [16, Lemma 5.9]), and thus if we define

$$n_\lambda = n_{\widehat{\chi}_\lambda} = \frac{|P\lambda|}{|P\lambda \cap \lambda P|} = \frac{|P\lambda P|}{|P\lambda|},$$

then since  $\rho_P = \sum_{\widehat{\phi} \in Irr(P)} \widehat{\phi}(1) \widehat{\phi}$  we conclude that

$$(3b) \quad n_\lambda \widehat{\chi}_\lambda = \sum_{\widehat{\phi} \in Irr_\lambda(P)} \widehat{\phi}(1) \widehat{\phi}$$

where  $\text{Irr}_\lambda(P)$  denotes the set consisting of all irreducible constituents of  $\widehat{\chi}_\lambda$ . In particular, it follows that every irreducible character of  $P$  is a constituent of a unique supercharacter. Therefore, in order to have a supercharacter theory, it remains to show that a supercharacter is indeed a character of  $P$ , and this is proved in [16, Theorems 5.4 and 5.6] (see also Section 4).

In order to define the supercharacters of the  $\sigma$ -fixed subgroup  $C_P(\sigma)$ , we consider  $\sigma$ -invariant supercharacters of  $P$ ; we observe that, if  $\text{cf}(P)$  denotes the complex vector space consisting of all class functions of  $P$ , then  $\sigma$  acts naturally on  $\text{cf}(P)$  by the rule  $\psi^\sigma(x) = \psi(x^\sigma)$  for all  $\psi \in \text{cf}(P)$  and all  $x \in P$ . For our purposes, it is convenient to define the supercharacters of  $P$  by means of the inverse Cayley transform  $\Psi: P \rightarrow \mathcal{J}$  as follows. For every  $\lambda \in \mathcal{J}^\circ$ , we define the function  $\widehat{\xi}_\lambda: P \rightarrow \mathbb{C}$  by the formula

$$(3c) \quad \widehat{\xi}_\lambda(x) = \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda P} \mu(\Psi(x))$$

for all  $x \in P$ . We have the following result (which allows us to use the word “supercharacter” when we refer to any of these functions).

**Proposition 3.1.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . Then, for every  $\lambda \in \mathcal{J}^\circ$  the supercharacter  $\widehat{\chi}_\lambda \in \text{SCh}(P)$  equals the function  $\widehat{\xi}_{\lambda^2}$ . In particular, we have  $\text{SCh}(P) = \{\widehat{\xi}_\lambda: \lambda \in \mathcal{J}^\circ\}$ .*

*Proof.* By Eq. (3a), we see that  $\widehat{\xi}_\lambda(\Phi(a)) = \widehat{\chi}_\lambda(1+a)$  for all  $a \in \mathcal{J}$ . In fact, since supercharacters are constant on superclasses, Eq. (1c) implies that

$$\begin{aligned} \widehat{\chi}_\lambda(\Phi(a)) &= \widehat{\chi}_\lambda(1+2a(1-a)^{-1}) = \widehat{\chi}_\lambda(1+2a) \\ &= \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda P} \mu(2a) = \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda P} \mu^2(a) \end{aligned}$$

for all  $a \in \mathcal{J}$ . Since  $(x\lambda y)^2 = x\lambda^2 y$  for all  $x, y \in P$  (as it is easily seen), it follows that

$$\widehat{\chi}_\lambda(\Phi(a)) = \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda^2 P} \mu(a) = \widehat{\xi}_{\lambda^2}(\Phi(a))$$

for all  $a \in \mathcal{J}$  as required.  $\square$

We next show that the  $\sigma$ -action on  $\text{cf}(P)$  restricts to a  $\sigma$ -action on  $\text{SCh}(P)$ . We first observe that

$$P\lambda^\sigma P = (P\lambda P)^\sigma = \{\mu^\sigma: \mu \in P\lambda P\};$$

in fact,  $(x\lambda y)^\sigma = y^{-\sigma}\lambda^\sigma x^{-\sigma}$  for all  $x, y \in P$ .

**Lemma 3.2.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , let  $P = 1 + \mathcal{J}$ , and let  $\lambda \in \mathcal{J}^\circ$ . Then,  $(\widehat{\xi}_\lambda)^\sigma = \widehat{\xi}_{\lambda^\sigma}$ , and thus  $(\widehat{\xi}_\lambda)^\sigma$  is a supercharacter of  $P$ .*

*Proof.* If  $x \in P$ , then  $\Psi(x)^\sigma = \Psi(x^\sigma)$ , and so we deduce that

$$\begin{aligned} (\widehat{\xi}_\lambda)^\sigma(x) &= \widehat{\xi}_\lambda(x^\sigma) = \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda P} \mu(\Psi(x^\sigma)) = \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda P} \mu(\Psi(x)^\sigma) \\ &= \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda P} \mu^\sigma(\Psi(x)) = \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda^\sigma P} \mu(\Psi(x)) = \widehat{\xi}_{\lambda^\sigma}(x) \end{aligned}$$

as required.  $\square$

We denote by  $\text{SCh}_\sigma(P)$  the subset of  $\text{SCh}(P)$  consisting of all  $\sigma$ -invariant supercharacters. The following result describes this subset; we recall that  $C_J(\sigma)^\circ = \{\lambda \in J^\circ : \lambda^\sigma = \lambda\}$ .

**Proposition 3.3.** *Let  $J$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + J$ . Then,  $\text{SCh}_\sigma(P) = \{\widehat{\xi}_\lambda : \lambda \in C_J(\sigma)^\circ\}$ .*

*Proof.* By the previous lemma, we see that  $(\widehat{\xi}_\lambda)^\sigma = \widehat{\xi}_{\lambda^\sigma} = \widehat{\xi}_\lambda$ , and thus  $\widehat{\xi}_\lambda \in \text{SCh}_\sigma(P)$  for all  $\lambda \in C_J(\sigma)^\circ$ . Conversely, let  $\mu \in J^\circ$  be such that  $\widehat{\xi}_\mu \in \text{SCh}_\sigma(P)$ . Since  $\widehat{\xi}_\mu = (\widehat{\xi}_\mu)^\sigma = \widehat{\xi}_{\mu^\sigma}$ , we conclude that  $\mu^\sigma \in P\mu P$ , and this clearly implies that the two-sided orbit  $P\mu P$  is  $\sigma$ -invariant. Now, we consider  $\sigma$  as the automorphism of  $P \times P$  given by  $(x, y)^\sigma = (y^\sigma, x^\sigma)$  for all  $x, y \in P$ , and observe that

$$((x, y) \cdot \nu)^\sigma = (x^{-1}\nu y)^\sigma = y^{-\sigma}\nu^\sigma x^\sigma = (y^\sigma, x^\sigma) \cdot \nu^\sigma = (x, y)^\sigma \cdot \nu^\sigma$$

for all  $x, y \in P$  and all  $\nu \in J^\circ$ . Thus, since  $P \times P$  acts transitively on  $P\mu P$  (and since  $2 \nmid |P|$ ), Glauberman's Lemma (see [20, Lemma 13.8]) implies that there exists  $\lambda \in P\mu P$  such that  $\lambda^\sigma = \lambda$ . Since  $\widehat{\xi}_\lambda = \widehat{\xi}_\mu$ , the result follows.  $\square$

As in the case of superclasses, it is natural to expect that supercharacters of  $C_P(\sigma)$  would be in one-to-one correspondence with  $\sigma$ -invariant two-sided orbits of  $P$  on  $J$ , and in fact we shall prove that a given supercharacter is determined by the subset consisting of the  $\sigma$ -fixed elements in the corresponding  $\sigma$ -invariant two-sided orbit. For any  $\lambda \in C_J(\sigma)^\circ$ , we define  $\Omega_P(\lambda)$  to be the subset of  $P\lambda P$  consisting of all  $\sigma$ -fixed elements.

**Proposition 3.4.** *Let  $J$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + J$ . Then,*

$$\Omega_P(\lambda) = \{x^{-1}\lambda x^\sigma : x \in P\}$$

for all  $\lambda \in C_J(\sigma)^\circ$ .

*Proof.* As before, we consider  $\sigma$  as an automorphism of  $P \times P$ . By [20, Corollary 13.9], the set of  $\sigma$ -fixed elements of  $P\lambda P$  is an orbit under the action of  $C_{P \times P}(\sigma)$ , and the result follows because  $C_{P \times P}(\sigma) = \{(x, x^\sigma) : x \in P\}$ .  $\square$

Next, we consider the *Glauberman correspondence* between  $\sigma$ -invariant irreducible characters of  $P$  and irreducible characters of  $C_P(\sigma)$ ; our main reference is [20, Chapter 13]. Since  $p$  is odd, this correspondence asserts that there exists a uniquely defined bijective map

$$\pi_P: \text{Irr}_\sigma(P) \rightarrow \text{Irr}(C_P(\sigma))$$

such that, for any  $\widehat{\chi} \in \text{Irr}_\sigma(P)$ , the image  $\chi = \pi_P(\widehat{\chi})$  is the unique irreducible constituent of the restriction  $\widehat{\chi}_{C_P(\sigma)}$  with odd multiplicity (see [20, Theorem 13.1]); here, and henceforth, we denote by  $\text{Irr}_\sigma(P)$  the subset of  $\text{Irr}(P)$  consisting of all  $\sigma$ -invariant irreducible characters of  $P$ .

**Lemma 3.5.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . Let  $\chi$  be any irreducible character of  $C_P(\sigma)$ , let  $\widehat{\chi} \in \text{Irr}_\sigma(P)$  be such that  $\pi_P(\widehat{\chi}) = \chi$ , and let  $\widehat{\xi} \in \text{SCh}(P)$  be the unique supercharacter such that  $\langle \widehat{\chi}, \widehat{\xi} \rangle \neq 0$ . Then,  $\widehat{\xi}^\sigma = \widehat{\xi}$ , and in particular there exists  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$  such that  $\widehat{\xi} = \widehat{\xi}_\lambda$ .*

*Proof.* This is an immediate consequence of the orthogonality of supercharacters because  $\widehat{\chi} = \widehat{\chi}^\sigma$  is an irreducible constituent of the supercharacter  $\widehat{\xi}^\sigma$  of  $P$ .  $\square$

For any  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$ , we write  $X(\lambda)$  to denote the set consisting of all irreducible characters  $\chi \in \text{Irr}(C_P(\sigma))$  such that Glauberman correspondent  $\widehat{\chi} \in \text{Irr}_\sigma(P)$  of  $\chi$  is a constituent of the supercharacter  $\widehat{\xi}_\lambda \in \text{SCh}_\sigma(P)$ , and define

$$(3d) \quad \sigma_\lambda = \sum_{\chi \in X(\lambda)} \chi(1)\chi;$$

notice that this is precisely the character  $\sigma_{X(\lambda)}$  of  $C_P(\sigma)$  defined in Eq. (1a).

**Theorem 3.6.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . Then,  $\{X(\lambda): \lambda \in C_{\mathcal{J}}(\sigma)^\circ\}$  is a partition of  $\text{Irr}(C_P(\sigma))$ ; in particular, every irreducible character  $\chi \in \text{Irr}(C_P(\sigma))$  is a constituent of  $\sigma_\lambda$  for some  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$ . Furthermore, for every  $\lambda, \mu \in C_{\mathcal{J}}(\sigma)^\circ$  we have  $\sigma_\lambda = \sigma_\mu$  if and only if  $\mu \in \Omega_P(\lambda)$ .*

*Proof.* By the previous lemma, it is clear that  $\text{Irr}(C_P(\sigma))$  is the union

$$\text{Irr}(C_P(\sigma)) = \bigcup_{\lambda \in C_{\mathcal{J}}(\sigma)^\circ} X(\lambda).$$

To show that this union is disjoint, let  $\chi \in X(\lambda) \cap X(\mu)$  for  $\lambda, \mu \in C_{\mathcal{J}}(\sigma)^\circ$ , and let  $\widehat{\chi} \in \text{Irr}_\sigma(P)$  be such that  $\chi = \pi_P(\widehat{\chi})$ . Then,  $\widehat{\chi}$  is a common irreducible constituent of the supercharacters  $\widehat{\xi}_\lambda, \widehat{\xi}_\mu \in \text{SCh}(P)$ , and thus  $\widehat{\xi}_\lambda = \widehat{\xi}_\mu$  (by the orthogonality of supercharacters). It follows that  $\mu \in P\lambda P$ , and the result is now a consequence of Eq. (1h) and Proposition 3.4.  $\square$

As a consequence of this theorem, we see that

$$X(\lambda) = \{\pi_P(\widehat{\chi}): \widehat{\chi} \in \text{Irr}_\sigma(P), \langle \widehat{\chi}, \widehat{\xi}_\lambda \rangle \neq 0\}$$

for all  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$ . Furthermore, the theorem suggests that, if  $\mathcal{X} = \{X(\lambda) : \lambda \in C_{\mathcal{J}}(\sigma)^\circ\}$  and  $\mathcal{K} = \text{SCl}(C_P(\sigma))$  (as in Eq. (2a)), then the pair  $(\mathcal{X}, \mathcal{K})$  forms a supercharacter theory for the  $\sigma$ -fixed subgroup  $C_P(\sigma)$ ; alternatively, we may define

$$(3e) \quad \text{SCh}(C_P(\sigma)) = \{\sigma_\lambda : \lambda \in C_{\mathcal{J}}(\sigma)^\circ\}$$

as the set of supercharacters of  $C_P(\sigma)$ . Further evidence is given by the following result.

**Theorem 3.7.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . Then, the sets*

- $\mathcal{X} = \{X(\lambda) : \lambda \in C_{\mathcal{J}}(\sigma)^\circ\}$ ,
- $\text{SCh}(C_P(\sigma)) = \{\sigma_\lambda : \lambda \in C_{\mathcal{J}}(\sigma)^\circ\}$ , and
- $\text{SCl}(C_P(\sigma)) = \{\widehat{\mathcal{K}} \cap C_P(\sigma) : \widehat{\mathcal{K}} \in \text{SCl}_\sigma(P)\}$

have the same cardinality.

*Proof.* By the previous theorem, it is obvious that  $|\mathcal{X}| = |\text{SCh}(C_P(\sigma))|$ . To prove of the other equality, we consider the action of  $P$  on  $C_{\mathcal{J}}(\sigma)$  given by  $x \cdot a = xax^{-\sigma}$  for all  $x \in P$  and all  $a \in C_{\mathcal{J}}(\sigma)$ , and denote by  $\Omega$  the set consisting of all orbits of  $P$  on  $C_{\mathcal{J}}(\sigma)$ ; notice that  $|\Omega| = |\text{SCl}(C_P(\sigma))|$  (by Theorem 2.4). On the other hand, we also consider the contragradient action of  $P$  on the dual group  $C_{\mathcal{J}}(\sigma)^\circ$  given by  $x \cdot \lambda = x\lambda x^{-\sigma}$  or all  $x \in P$  and all  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$ , and denote by  $\Omega^\circ$  the set consisting of all orbits of  $P$  on  $C_{\mathcal{J}}(\sigma)^\circ$ . By Theorem 3.6 we have  $|\Omega^\circ| = |\mathcal{X}|$ , and thus we must prove that  $|\Omega| = |\Omega^\circ|$ . To see this, let  $\tau$  be the permutation character of  $P$  on  $C_{\mathcal{J}}(\sigma)$ ; hence,  $\tau(x) = |\{a \in C_{\mathcal{J}}(\sigma) : x \cdot a = a\}|$  for all  $x \in P$ . Since  $(x \cdot \lambda)(x \cdot a) = \lambda(a)$  for all  $x \in P$ , all  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$  and all  $a \in C_{\mathcal{J}}(\sigma)$ , it follows from Brauer's Theorem ([20, Theorem 6.32]) that  $\tau(x) = |\{\lambda \in C_{\mathcal{J}}(\sigma)^\circ : x \cdot \lambda = \lambda\}|$  for all  $x \in P$ , and thus  $\tau$  is also the permutation character of  $P$  on  $C_{\mathcal{J}}(\sigma)^\circ$ . By [20, Corollary 5.15], we conclude that  $|\Omega| = \langle \tau, 1_P \rangle = |\Omega^\circ|$  as required.  $\square$

Thus, in order to establish that we have a genuine supercharacter theory for  $C_P(\sigma)$  only one thing remains: we must show that for every  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$ , the (super)character  $\sigma_\lambda$  is a superclass function. This will be a consequence of the following main result which gives a convenient way to compute the values of a supercharacter.

**Theorem 3.8.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , let  $P = 1 + \mathcal{J}$ , and let  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$ . Then,*

$$(3f) \quad \sigma_\lambda(x) = \sum_{\mu \in \Omega_P(\lambda)} \mu(\Psi(x))$$

for all  $x \in C_P(\sigma)$ . In particular,  $\sigma_\lambda$  has a constant value on each superclass of  $C_P(\sigma)$ .

The proof of this theorem will be the main goal of the next two sections. Once the theorem is proved, then we can define for every  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$  the *supercharacter* of  $C_P(\sigma)$  associated with  $\lambda$  to be the function  $\varsigma_\lambda: C_P(\sigma) \rightarrow \mathbb{C}$  by the rule

$$(3g) \quad \varsigma_\lambda(x) = \sum_{\mu \in \Omega_P(\lambda)} \mu(\Psi(x))$$

for all  $x \in C_P(\sigma)$ ; notice that  $\varsigma_\lambda$  depends only on the orbit  $\Omega_P(\lambda) = \{x^{-1}\lambda x^\sigma : x \in P\}$  where we consider the action of  $P$  on the left of  $C_{\mathcal{J}}(\sigma)^\circ$  given by  $x \cdot \lambda = x^{-1}\lambda x^\sigma$  for all  $x \in P$  and all  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$ . It is clear that  $\varsigma_\lambda(yxy^{-1}) = \varsigma_\lambda(x)$  for all  $x, y \in C_P(\sigma)$ , and hence  $\varsigma_\lambda$  is a class function of  $C_P(\sigma)$ . Since  $\text{Irr}(C_P(\sigma))$  is a  $\mathbb{C}$ -basis of  $\text{cf}(C_P(\sigma))$ , it follows that  $\varsigma_\lambda$  is a  $\mathbb{C}$ -linear combination of the irreducible characters of  $C_P(\sigma)$ . Our aim is to prove that  $\varsigma_\lambda$  is a character of  $C_P(\sigma)$ , and this occurs if and only if it is a linear combination of irreducible characters with positive integer coefficients. In fact, Theorem 3.8 claims that  $\varsigma_\lambda = \sigma_\lambda = \sum_{\chi \in X(\lambda)} \chi(1)\chi$ , and thus we must prove that an irreducible character  $\chi \in \text{Irr}(C_P(\sigma))$  appears in the class function  $\varsigma_\lambda \in \text{cf}(C_P(\sigma))$  (with non-zero multiplicity) if and only if its Glauberman correspondent  $\widehat{\chi} \in \text{Irr}_\sigma(P)$  appears in the supercharacter  $\widehat{\xi}_\lambda \in \text{SCh}(P)$  (with non-zero multiplicity); moreover, if this is the case, then we must also show that the multiplicity  $\langle \chi, \sigma_\lambda \rangle$  equals the degree  $\chi(1)$  of  $\chi$ . To achieve this, we recall that by Eq. (3b) (see also [16, Theorem 5.5(ii)] and Proposition 3.1) we have

$$n_\lambda \widehat{\xi}_\lambda = \sum_{\widehat{\chi} \in \text{Irr}_\lambda(P)} \widehat{\chi}(1) \widehat{\chi}$$

where  $n_\lambda = |P\lambda P|/|P\lambda|$ ; furthermore, it follows from [16, Theorem 5.6] (and from Proposition 3.1) that  $n_\lambda \widehat{\xi}_\lambda = \widehat{\varsigma}_\lambda$  where  $\widehat{\varsigma}_\lambda: P \rightarrow \mathbb{C}$  is the function defined by the rule

$$(3h) \quad \widehat{\varsigma}_\lambda(x) = \sum_{\mu \in P\lambda P} \mu(\Psi(x))$$

for all  $x \in P$ . On the other hand, if  $\widehat{\chi} \in \text{Irr}_\sigma(\lambda)$  is an arbitrary  $\sigma$ -invariant irreducible constituent of  $\widehat{\xi}_\lambda$ , then [5, Theorem 2.1] asserts that there exist a  $\sigma$ -invariant algebra subgroup  $Q$  of  $P$  and a  $\sigma$ -invariant linear character  $\widehat{\vartheta} \in \text{Irr}_\sigma(Q)$  such that  $\widehat{\chi} = \widehat{\vartheta}^P$  and  $\chi = \vartheta^{C_P(\sigma)}$  where  $\chi = \pi_P(\widehat{\chi}) \in \text{Irr}(C_P(\sigma))$  and  $\vartheta = \pi_Q(\widehat{\vartheta}) \in \text{Irr}(C_Q(\sigma))$ ; given any  $\sigma$ -invariant subgroup  $Q$  of  $P$ , we write  $\pi_Q$  to denote the Glauberman map  $\pi_Q: \text{Irr}_\sigma(Q) \rightarrow \text{Irr}(C_Q(\sigma))$ . By the above (and by Frobenius reciprocity), we have  $\widehat{\chi}(1) = \langle \chi, n_\lambda \widehat{\xi}_\lambda \rangle = \langle \widehat{\vartheta}^P, \widehat{\varsigma}_\lambda \rangle = \langle \widehat{\vartheta}, (\widehat{\varsigma}_\lambda)_Q \rangle$ . By [16, Theorem 6.4], the restriction  $(\widehat{\xi}_\lambda)_Q$  decomposes as a sum of supercharacters of  $Q$ , and hence  $(\widehat{\varsigma}_\lambda)_Q = n_\lambda (\widehat{\xi}_\lambda)_Q$  also decomposes as a sum of supercharacters of  $Q$ . It follows that there exists a unique supercharacter  $\widehat{\xi}_0 \in \text{SCh}(Q)$  such that  $\widehat{\xi}_0$  is a constituent of  $(\widehat{\varsigma}_\lambda)_Q$  and  $\widehat{\vartheta}$  is a constituent of  $\widehat{\xi}_0$ . In light of this reduction process, we will first prove Theorem 3.8 in the more favourable situation where the supercharacter  $\widehat{\xi}_\lambda$  has a linear constituent.

#### 4. SUPERCHARACTERS WITH A LINEAR CONSTITUENT

As before, let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . Our aim is to prove Theorem 3.8 in the particular situation where  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$  is such that  $\widehat{\xi}_\lambda \in \text{SCh}(P)$  has a linear constituent. We start by recalling some general facts about the supercharacter  $\widehat{\xi}_\lambda$ . We define

$$\mathcal{L}(\lambda) = \{a \in \mathcal{J}: a\mathcal{J} \subseteq \ker(\lambda)\} \quad \text{and} \quad L(\lambda) = 1 + \mathcal{L}(\lambda).$$

Then,  $\mathcal{L}(\lambda)$  is a right ideal (hence, a subalgebra) of  $\mathcal{J}$ , and thus  $L(\lambda)$  is an algebra subgroup of  $P$ ; notice that  $L(\lambda) = \{x \in P: x\lambda = \lambda\}$  is the centralizer of  $\lambda$  with respect to the left action of  $P$  on  $\mathcal{J}$ . The mapping  $x \mapsto \lambda(x - 1)$  clearly defines a linear character  $\widehat{\tau}_\lambda: L(\lambda) \rightarrow \mathbb{C}^\times$ , and it is proved in [16, Theorems 5.4 and 5.6] that  $\widehat{\chi}_\lambda = (\widehat{\tau}_\lambda)^P$ ; recall that we are writing  $\widehat{\chi}_\lambda$  for the (super)character of  $P$  defined by Eq. (3a). [In particular, we conclude that  $\widehat{\chi}_\lambda$  is indeed a character of  $P$ .] Next, we prove that the supercharacter  $\widehat{\xi}_\lambda$  is also induced from a linear character of the subgroup  $L(\lambda)$ . In fact, since the Cayley transform  $\Phi: \mathcal{J} \rightarrow P$  clearly maps  $\mathcal{L}(\lambda)$  to  $L(\lambda)$  bijectively, we may define the function  $\widehat{\vartheta}_\lambda: L(\lambda) \rightarrow \mathbb{C}^\times$  by the rule

$$(4a) \quad \widehat{\vartheta}_\lambda(x) = \lambda(\Psi(x))$$

for all  $x \in L(\lambda)$ . Then, we obtain the following result (where we are not assuming that the supercharacter  $\widehat{\xi}_\lambda \in \text{SCh}(P)$  has a linear constituent).

**Proposition 4.1.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . Then, for every  $\lambda \in \mathcal{J}^\circ$  the function  $\widehat{\vartheta}_\lambda$  is a linear character of  $L(\lambda)$ , and we have  $\widehat{\xi}_\lambda = (\widehat{\vartheta}_\lambda)^P$ .*

*Proof.* By the definition of  $\mathcal{L}(\lambda)$ , it is clear that  $\lambda(ax) = \lambda(a)$  for all  $a \in \mathcal{L}(\lambda)$  and all  $x \in P$ . On the other hand, let  $\mu \in \mathcal{J}^\circ$  be such that  $\lambda = \mu^2$ . Then,  $\lambda(a) = \mu(2a)$  for all  $a \in \mathcal{J}$ , and thus  $\mu(ax) = \mu(a)$  for all  $a \in \mathcal{L}(\lambda)$  and all  $x \in P$ ; in fact, we have  $\mathcal{L}(\mu) = \mathcal{L}(\lambda)$ . In particular, we deduce that

$$\widehat{\vartheta}_\lambda(\Phi(a)) = \lambda(a) = \mu(2a) = \mu(2a(1 - a)^{-1}) = \mu(\Phi(a) - 1)$$

for all  $a \in \mathcal{L}(\lambda)$ , and thus  $\widehat{\vartheta}_\lambda(x) = \widehat{\tau}_\mu(x)$  for all  $x \in L(\lambda)$ . It follows that  $\widehat{\vartheta}_\lambda$  is a linear character of  $L(\lambda)$ , and that  $(\widehat{\vartheta}_\lambda)^P = (\widehat{\tau}_\mu)^P = \widehat{\chi}_\mu = \widehat{\xi}_\lambda$  (by Proposition 3.1).  $\square$

Under our assumption that  $\widehat{\xi}_\lambda \in \text{SCh}(P)$  has a linear constituent, [16, Corollary 5.12] assures that  $\mathcal{L}(\lambda)$  is a two-sided ideal of  $\mathcal{J}$ , and hence  $L(\lambda)$  is a normal subgroup of  $P$ ; furthermore, we have  $P\lambda = \lambda P = P\lambda P$ , and thus

$$\widehat{\xi}_\lambda(x) = \widehat{\varsigma}_\lambda(x) = \sum_{\mu \in P\lambda P} \mu(\Psi(x))$$

for all  $x \in \mathcal{J}$ . On the other hand, we observe that the subgroup  $L(\lambda)$  is  $\sigma$ -invariant: in fact, since  $\lambda$  is  $\sigma$ -invariant, we have  $(x\lambda)^\sigma = \lambda x^\sigma$  for all  $x \in P$ , and thus  $L(\lambda)^\sigma = L(\lambda)$

(again by [16, Corollary 5.12]). We now consider the  $\sigma$ -fixed subgroup  $C_{L(\lambda)}(\sigma)$ , and note that  $C_{L(\lambda)}(\sigma) = \Phi(C_{\mathcal{L}(\lambda)}(\sigma))$  where  $C_{\mathcal{L}(\lambda)}(\sigma) = \{a \in C_{\mathcal{J}}(\sigma) : a^\sigma = a\}$ . We define the linear character  $\vartheta_\lambda : C_{L(\lambda)}(\sigma) \rightarrow \mathbb{C}^\times$  to be the restriction of  $\widehat{\vartheta}_\lambda$  to  $C_{L(\lambda)}(\sigma)$ ; hence,

$$(4b) \quad \vartheta_\lambda(x) = \lambda(\Psi(x))$$

for all  $x \in C_{L(\lambda)}(\sigma)$ . Furthermore, we define  $\xi_\lambda$  to be the induced character

$$(4c) \quad \xi_\lambda = (\vartheta_\lambda)^{C_P(\sigma)}.$$

The following result is a simple consequence of [20, Theorem 13.29]; we recall that  $L(\lambda)$  is a normal subgroup of  $P$ .

**Lemma 4.2.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , let  $P = 1 + \mathcal{J}$ , and let  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$  be such that the supercharacter  $\widehat{\xi}_\lambda \in \text{SCh}(P)$  has a linear constituent. Let  $\chi \in \text{Irr}(C_P(\sigma))$ , and let  $\widehat{\chi} \in \text{Irr}_\sigma(P)$  be such that  $\pi_P(\widehat{\chi}) = \chi$ . Then,  $\langle \chi, \xi_\lambda \rangle \neq 0$  if and only if  $\langle \widehat{\chi}, \widehat{\xi}_\lambda \rangle \neq 0$ .*

We are now able to prove the following particular case of Theorem 3.8.

**Theorem 4.3.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , let  $P = 1 + \mathcal{J}$ , and let  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$  be such that the supercharacter  $\widehat{\xi}_\lambda \in \text{SCh}(P)$  has a linear constituent. Then,*

$$\sigma_\lambda(x) = \sum_{\mu \in \Omega_P(\lambda)} \mu(\Psi(x))$$

for all  $x \in C_P(\sigma)$ . Moreover, we have  $\sigma_\lambda = \xi_\lambda = (\vartheta_\lambda)^{C_P(\sigma)}$ .

*Proof.* Let  $x \in C_P(\sigma)$  be arbitrary. We show that both members of the desired equality are equal to 0 unless  $x \in C_{L(\lambda)}(\sigma)$  in which case they are both equal to  $|C_P(\sigma) : C_{L(\lambda)}(\sigma)| \vartheta_\lambda(x)$ .

To start with, we observe that this is precisely the value  $\xi_\lambda(x) = (\vartheta_\lambda)^{C_P(\sigma)}(x)$ . In fact, since  $L(\lambda)$  is a normal subgroup of  $P$ ,  $C_{L(\lambda)}(\sigma) = L(\lambda) \cap C_P(\sigma)$  is a normal subgroup of  $C_P(\sigma)$ , and thus  $(\vartheta_\lambda)^{C_P(\sigma)}(x) = 0$  whenever  $x \notin C_{L(\lambda)}(\sigma)$ . On the other hand, by [10, Corollary 4.3] the linear character  $\widehat{\vartheta}_\lambda : L(\lambda) \rightarrow \mathbb{C}^\times$  is  $P$ -invariant, and so its restriction to  $C_{L(\lambda)}(\sigma)$  is  $C_P(\sigma)$ -invariant. It follows that  $(\vartheta_\lambda)^{C_P(\sigma)}(x) = |C_P(\sigma) : C_{L(\lambda)}(\sigma)| \vartheta_\lambda(x)$  whenever  $x \in C_{L(\lambda)}(\sigma)$ .

Next, we show that

$$(4d) \quad \xi_\lambda(x) = \sum_{\mu \in \Omega_P(\lambda)} \mu(\Psi(x)).$$

By [16, Lemma 4.2], we have  $P\lambda = \lambda + \mathcal{L}(\lambda)^\perp$  where  $\mathcal{L}(\lambda)^\perp = \{\nu \in \mathcal{J}^\circ : \mathcal{L}(\lambda) \subseteq \ker(\nu)\}$ . Since  $\Omega_P(\lambda) = \{\mu \in P\lambda P : \mu^\sigma = \mu\}$  (by Proposition 3.4) and since  $P\lambda P = P\lambda = \lambda P$ , we conclude that  $\Omega_P(\lambda) = \lambda + \{\nu \in \mathcal{L}(\lambda)^\perp : \nu^\sigma = \nu\}$ . If  $\nu \in \mathcal{J}^\circ$ , then  $\nu^\sigma = \nu$  if and only if  $\nu \in C_{\mathcal{J}}(\sigma)^\circ$ , and so  $\{\nu \in \mathcal{L}(\lambda)^\perp : \nu^\sigma = \nu\} = C_{\mathcal{J}}(\sigma)^\circ \cap \mathcal{L}(\lambda)^\perp$ ; moreover,  $\mathcal{L}(\lambda) \subseteq \ker(\nu)$  if and

only if  $C_{\mathcal{L}(\lambda)}(\sigma) \subseteq \ker(\nu)$ , and thus  $C_{\mathcal{J}}(\sigma)^\circ \cap \mathcal{L}(\lambda)^\perp = C_{\mathcal{L}(\lambda)}(\sigma)^\perp$  where we set  $C_{\mathcal{L}(\lambda)}(\sigma)^\perp = \{\nu \in C_{\mathcal{J}}(\sigma)^\circ : C_{\mathcal{L}(\lambda)}(\sigma) \subseteq \ker(\nu)\}$ . It follows that  $\Omega_P(\lambda) = \lambda + C_{\mathcal{L}(\lambda)}(\sigma)^\perp$ , and thus

$$\sum_{\mu \in \Omega_P(\lambda)} \mu(a) = \lambda(a) \sum_{\nu \in C_{\mathcal{L}(\lambda)}(\sigma)^\perp} \nu(a)$$

where  $a = \Psi(x) \in C_{\mathcal{J}}(\sigma)$ . Since the sum  $\sum_{\nu \in C_{\mathcal{L}(\lambda)}(\sigma)^\perp} \nu$  naturally identifies with the regular character of the additive group  $C_{\mathcal{J}}(\sigma)/C_{\mathcal{L}(\lambda)}(\sigma)$ , we conclude that

$$\sum_{\mu \in \Omega_P(\lambda)} \mu(a) = \begin{cases} 0, & \text{if } a \notin C_{\mathcal{L}(\lambda)}(\sigma), \\ |C_{\mathcal{J}}(\sigma) : C_{\mathcal{L}(\lambda)}(\sigma)| \lambda(a), & \text{if } a \in C_{\mathcal{L}(\lambda)}(\sigma), \end{cases}$$

and Eq. (4d) follows because the Cayley transform  $\Phi: \mathcal{J} \rightarrow P$  is bijective and maps  $C_{\mathcal{J}}(\sigma)$  to  $C_P(\sigma)$  and  $C_{\mathcal{L}(\lambda)}(\sigma)$  to  $C_{L(\lambda)}(\sigma)$ .

To conclude the proof, we apply Gallagher's Theorem (see [20, Corollary 6.17]) to identify the irreducible constituents of  $\xi_\lambda = (\vartheta_\lambda)^{C_P(\lambda)}$ ; we recall that  $C_{L(\lambda)}(\sigma)$  is a normal subgroup of  $C_P(\sigma)$ . We first claim that the linear character  $\vartheta_\lambda$  of  $C_{L(\lambda)}(\sigma)$  extends to  $C_P(\sigma)$ . To see this, let  $\widehat{\tau} \in \text{Irr}(P)$  be a linear constituent of  $\widehat{\xi}_\lambda$  (which exists by assumption), and let  $\tau$  be its restriction to  $C_P(\sigma)$ . (Notice that  $\widehat{\tau}$  is not necessarily  $\sigma$ -invariant, hence it may not be the Glauberman correspondent of  $\tau$ .) Since  $\widehat{\vartheta}_\lambda$  is  $P$ -invariant, we have  $\widehat{\tau}_{L(\lambda)} = \widehat{\vartheta}_\lambda$ , and hence

$$\tau_{C_{L(\lambda)}(\sigma)} = (\widehat{\vartheta}_\lambda)_{C_{L(\lambda)}(\sigma)} = \vartheta_\lambda.$$

Therefore,  $\tau$  is an extension of  $\vartheta_\lambda$  to  $C_P(\sigma)$ , and so Gallagher's Theorem implies that

$$\xi_\lambda = (\vartheta_\lambda)^{C_P(\sigma)} = \sum_{\substack{\omega \in \text{Irr}(C_P(\sigma)) \\ C_{L(\lambda)}(\sigma) \subseteq \ker(\omega)}} \omega(1)(\tau\omega).$$

Finally, it easily seen from Proposition 3.3 that

$$X(\lambda) = \{\tau\omega : \psi \in \text{Irr}(C_P(\sigma)), C_{L(\lambda)}(\sigma) \subseteq \ker(\psi)\},$$

and thus

$$\xi_\lambda = \sum_{\chi \in X(\lambda)} \chi(1)\chi = \sigma_\lambda.$$

The proof is complete.  $\square$

## 5. PROOF OF THEOREM 3.8

Let  $\mathcal{J}$  a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . Otherwise stated, we fix a linear character  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$  throughout the section. Our primary goal is to show that Eq. (3f) holds, and we shall use the reduction process described before. We let  $\widehat{\chi} \in \text{Irr}_\sigma(\lambda)$  be an

arbitrary  $\sigma$ -invariant irreducible constituent of the supercharacter  $\widehat{\xi}_\lambda \in \text{SCh}(P)$ , and choose a  $\sigma$ -invariant algebra subgroup  $Q$  of  $P$  and a  $\sigma$ -invariant linear character  $\widehat{\vartheta}$  of  $Q$  such that

$$\widehat{\chi} = \widehat{\vartheta}^P \quad \text{and} \quad \chi = \vartheta^{C_P(\sigma)}$$

where  $\chi = \pi_P(\widehat{\chi}) \in \text{Irr}(C_P(\sigma))$  and  $\vartheta = \pi_Q(\widehat{\vartheta}) \in \text{Irr}(C_Q(\sigma))$  (the existence of  $Q$  and  $\widehat{\vartheta}$  is guaranteed by [5, Theorem 2.1]). Then,  $\widehat{\chi}(1) = \langle \chi, \widehat{\varsigma}_\lambda \rangle = \langle \widehat{\vartheta}, (\widehat{\varsigma}_\lambda)_Q \rangle$ , and thus there exists a unique supercharacter  $\widehat{\xi}_0 \in \text{SCh}(Q)$  such that  $\widehat{\xi}_0$  is a constituent of  $(\widehat{\varsigma}_\lambda)_Q$  and  $\widehat{\vartheta}$  is a constituent of  $\widehat{\xi}_0$ ; recall that the restriction  $(\widehat{\varsigma}_\lambda)_Q = n_\lambda (\widehat{\xi}_\lambda)_Q$  decomposes as a sum of supercharacters of  $Q$  (by [16, Theorem 6.4]). We now prove the following result (which holds for every algebra group).

**Proposition 5.1.** *Let  $P = 1 + \mathcal{J}$  be an algebra group over  $\mathbb{k}$ , and let  $\widehat{\chi} \in \text{Irr}(P)$  be an irreducible constituent of a supercharacter  $\widehat{\xi} \in \text{SCh}(P)$ . Let  $\mathcal{I}$  be a subalgebra of  $\mathcal{J}$ , let  $Q = 1 + \mathcal{I}$ , and suppose that  $\widehat{\chi} = \widehat{\vartheta}^P$  for some a linear character  $\widehat{\vartheta}$  of  $Q$ . Let  $\widehat{\xi}_0 \in \text{SCh}(Q)$  be the unique supercharacter of  $Q$  such that  $\widehat{\vartheta}$  is a constituent of  $\widehat{\xi}_0$ . Then:*

- (1)  *$\widehat{\xi}_0$  is a constituent of the restriction  $\widehat{\xi}_Q$  with multiplicity  $\widehat{\chi}(1)$ .*
- (2) *There exists  $\lambda \in \mathcal{J}^\circ$  such that  $\widehat{\xi} = \widehat{\xi}_\lambda$  and  $\widehat{\xi}_0 = \widehat{\xi}_{\lambda_0}$  where  $\lambda_0 = \lambda|_{\mathcal{I}}$  is the restriction of  $\lambda$  to  $\mathcal{I}$ .*
- (3) *If  $\mu_0 \in Q\lambda_0 Q$  and  $\mu \in \mathcal{J}^\circ$  is such that  $\mu|_{\mathcal{I}} = \mu_0$ , then  $\mu + \mathcal{J}^\perp \subseteq P\lambda P$ ; in particular, the set  $\{\mu \in P\lambda P : \mu|_{\mathcal{I}} = \mu_0\}$  has cardinality  $|\mathcal{J} : \mathcal{I}| = |P : Q|$ .*

*Proof.* Since  $\widehat{\vartheta}$  is linear, [16, Corollary 5.12] asserts that  $Q\lambda_0 = \lambda_0 Q = Q\lambda_0 Q$ , and thus

$$\widehat{\xi}_0 = \sum_{\widehat{\chi}_0 \in \text{Irr}_{\lambda_0}(Q)} \widehat{\chi}_0(1) \widehat{\chi}_0$$

(by [16, Theorem 5.5(ii)]). Since  $\widehat{\chi}(1) = \langle \widehat{\vartheta}, (\widehat{\varsigma}_\lambda)_Q \rangle$  and  $\langle \widehat{\vartheta}, \widehat{\xi}_0 \rangle = \widehat{\vartheta}(1) = 1$ , we conclude that

$$(\widehat{\varsigma}_\lambda)_Q = \widehat{\chi}(1) \widehat{\xi}_0 + \widehat{\zeta}$$

where  $\widehat{\zeta}$  is a sum of supercharacters of  $Q$  all distinct from  $\widehat{\xi}_0$ ; in particular, we have  $\langle \widehat{\vartheta}, \widehat{\zeta} \rangle = 0$ . By the definition of  $\widehat{\varsigma}_\lambda$  and of  $\widehat{\xi}_0$  (see Eq. (3c)), we deduce that

$$\sum_{\mu \in P\lambda P} \mu|_{\mathcal{I}} = \widehat{\chi}(1) \sum_{\mu_0 \in Q\lambda_0 Q} \mu_0 + \mu'$$

where  $\mu'$  is a character (not necessarily linear) of the additive group  $\mathcal{J}^+$  satisfying  $\langle \mu', \mu_0 \rangle = 0$  for all  $\mu_0 \in Q\lambda_0 Q$ . It follows that every linear character  $\mu_0 \in Q\lambda_0 Q$  occurs with multiplicity  $\widehat{\chi}(1)$  in the sum of the left hand side, and hence the set  $\{\mu \in P\lambda P : \mu|_{\mathcal{I}} = \mu_0\}$  has cardinality  $\widehat{\chi}(1)$ . Since

$$\widehat{\chi}(1) = \widehat{\vartheta}^P(1) = |P : Q| = |\mathcal{J} : \mathcal{I}| = |\mathcal{J}^\perp|,$$

we conclude that  $\mu + \mathcal{I}^\perp \subseteq P\lambda P$  for all  $\mu \in P\lambda P$  such that  $\mu_{\mathcal{J}} \in Q\lambda_0 Q$ , and this completes the proof.  $\square$

We are now able to proceed with the proof of Theorem 3.8.

*Proof of Theorem 3.8.* We assume that  $\lambda \in C_{\mathcal{J}}(\sigma)^\circ$ , and let the notation be as above; without loss of generality, we may assume that  $\widehat{\xi}_0 = \widehat{\xi}_{\lambda_0}$  is the supercharacter of  $Q$  corresponding to the restriction  $\lambda_0 = \lambda_{\mathcal{J}}$  of  $\lambda$  to  $\mathcal{J} = Q - 1$ . Let  $\Omega_Q(\lambda_0) = \{x^{-1}\lambda_0 x^\sigma : x \in Q\}$  be the subset of  $Q\lambda_0 Q \subseteq \mathcal{J}^\circ$  consisting of  $\sigma$ -fixed elements, and consider the function  $\varsigma_0 : C_Q(\sigma) \rightarrow \mathbb{C}$  given by

$$\varsigma_0(x) = \sum_{\mu_0 \in \Omega_Q(\lambda_0)} \mu_0(\Psi(x))$$

for all  $x \in C_Q(\sigma)$ . Then, since  $\widehat{\vartheta} \in \text{Irr}_\sigma(Q)$  is a  $\sigma$ -invariant linear constituent of the supercharacter  $\widehat{\xi}_0 \in \text{SCh}(Q)$ , Theorem 4.3 implies that

$$\varsigma_0 = \sum_{\chi_0 \in X(\lambda_0)} \chi_0(1)\chi_0$$

where  $X(\lambda_0) = \{\pi_Q(\widehat{\chi}_0) : \widehat{\chi}_0 \in \text{Irr}_\sigma(Q), \langle \widehat{\chi}_0, \widehat{\xi}_0 \rangle \neq 0\}$ ; in particular,  $\vartheta = \pi_Q(\widehat{\vartheta}) \in \text{Irr}(C_Q(\sigma))$  is a linear constituent of  $\varsigma_0$  occurring with multiplicity one. Our goal is to show that the irreducible character  $\chi = \vartheta^{C_P(\sigma)}$  appears as a constituent of  $\varsigma_\lambda$  with multiplicity

$$\chi(1) = |C_P(\sigma) : C_Q(\sigma)| = |C_{\mathcal{J}}(\sigma) : C_{\mathcal{J}}(\sigma)| = |C_{\mathcal{J}}(\sigma)^\perp|$$

where  $C_{\mathcal{J}}(\sigma)^\perp = \{\nu \in C_{\mathcal{J}}(\sigma)^\circ : C_{\mathcal{J}}(\sigma) \subseteq \ker(\nu)\}$ .

Firstly, observe that Theorem 2.4 and Proposition 3.4 clearly imply that for all  $\nu_0 \in C_{\mathcal{J}}(\sigma)^\circ$  the function  $\varsigma_{\nu_0} : C_Q(\sigma) \rightarrow \mathbb{C}$  (defined as in Eq. (3g)) is constant on each superclass of  $C_Q(\sigma)$ ; moreover, the proof of Proposition 3.4 shows that  $\{\Omega_Q(\nu_0) : \nu_0 \in C_{\mathcal{J}}(\sigma)^\circ\}$  is a partition of  $C_{\mathcal{J}}(\sigma)^\circ$ . It follows that  $\{\varsigma_{\nu_0} : \nu_0 \in C_{\mathcal{J}}(\sigma)^\circ\}$  is an orthogonal basis of the complex space  $\text{scf}(C_Q(\sigma))$  consisting of all superclass functions of  $C_Q(\sigma)$ . Therefore, since the restriction  $(\varsigma_\lambda)_{C_Q(\sigma)}$  of  $\varsigma_\lambda$  to  $C_Q(\sigma)$  is clearly a superclass function on  $C_Q(\sigma)$ , we conclude that there exist  $\nu_1, \dots, \nu_r \in C_{\mathcal{J}}(\sigma)^\circ$  and  $z_1, \dots, z_r \in \mathbb{C}$  such that  $(\varsigma_\lambda)_{C_Q(\sigma)} = z_1\varsigma_{\nu_1} + \dots + z_r\varsigma_{\nu_r}$  where  $\langle \varsigma_{\nu_i}, \varsigma_{\nu_j} \rangle = 0$  for all  $1 \leq i \neq j \leq r$ ; in other words, we have

$$\sum_{\mu \in \Omega_P(\lambda)} \mu_{C_{\mathcal{J}}(\sigma)} = z_1 \sum_{\mu_1 \in \Omega_Q(\nu_1)} (\mu_1)_{C_{\mathcal{J}}(\sigma)} + \dots + z_r \sum_{\mu_r \in \Omega_Q(\nu_r)} (\mu_r)_{C_{\mathcal{J}}(\sigma)}$$

where the  $Q$ -orbits  $\Omega_Q(\nu_1), \dots, \Omega_Q(\nu_r)$  are all distinct. In particular, we deduce that

$$z_i = |\{\mu \in \Omega_P(\sigma) : \mu_{C_{\mathcal{J}}(\sigma)} = \nu_i\}|$$

for all  $1 \leq i \leq r$ , and hence  $z_1, \dots, z_r$  are positive integers. Since  $\lambda_{C_J(\sigma)} = (\lambda_0)_{C_J(\sigma)} \in \Omega_Q(\nu_i)$  for some  $1 \leq i \leq r$ , we conclude that

$$(\varsigma_\lambda)_{C_Q(\sigma)} = m\varsigma_0 + \zeta$$

where  $m = |\{\mu \in \Omega_P(\sigma) : \mu_{C_J(\sigma)} = \lambda_{C_J(\sigma)}\}|$  and  $\zeta : C_Q(\sigma) \rightarrow \mathbb{C}$  is a superclass function satisfying  $\langle \varsigma_0, \zeta \rangle = 0$ ; moreover, since  $\vartheta \in \text{Irr}(C_Q(\sigma))$  is a linear constituent of  $\varsigma_0$ , Theorem 4.3 implies that  $\langle \vartheta, \zeta \rangle = 0$ . It follows that

$$m = \langle \vartheta, \varsigma_0 \rangle = \langle \vartheta, (\varsigma_\lambda)_{C_Q(\sigma)} \rangle = \langle \vartheta^{C_P(\sigma)}, \varsigma_\lambda \rangle = \langle \chi, \varsigma_\lambda \rangle,$$

and hence our claim is equivalent to showing that

$$|C_J(\sigma)^\perp| = \chi(1) = m = |\{\mu \in \Omega_P(\sigma) : \mu_{C_J(\sigma)} = \lambda_{C_J(\sigma)}\}|.$$

Since the mapping  $\mu \mapsto \mu_{C_J(\sigma)}$  defines a bijection  $\pi_J : \{\mu \in J^\circ : \mu^\sigma = \mu\} \rightarrow C_J(\sigma)^\circ$ , it also defines a bijection  $\pi_J : \{\mu \in \lambda + J^\perp : \mu^\sigma = \mu\} \rightarrow \lambda_{C_J(\sigma)} + C_J(\sigma)^\perp$ ; we recall that  $J$  is  $\sigma$ -invariant. Since  $\lambda + J^\perp \subseteq P\lambda P$  (by Proposition 5.1), we have  $\{\mu \in \lambda + J^\perp : \mu^\sigma = \mu\} = \Omega_P(\lambda) \cap (\lambda + J^\perp)$ , and thus

$$|C_J(\sigma)^\perp| = |\Omega_P(\lambda) \cap (\lambda + J^\perp)| = |\{\mu \in \Omega_P(\lambda) : \mu_J = \lambda_J\}|.$$

On the other hand, the bijection  $\pi_J : \{\mu \in J^\circ : \mu^\sigma = \mu\} \rightarrow C_J(\sigma)^\circ$  also gives

$$\{\mu \in \Omega_P(\lambda) : \mu_J = \lambda_J\} = \{\mu \in \Omega_P(\lambda) : \mu_{C_J(\sigma)} = \lambda_{C_J(\sigma)}\},$$

and thus we conclude that  $m = |C_J(\sigma)^\perp| = \chi(1)$ , as required. This concludes the proof of Theorem 3.8.  $\square$

Before we close this section, we give a brief summary of the principal results we obtained so far. Given a  $\sigma$ -invariant algebra subgroup  $P = 1 + J$  of  $\mathcal{A}^\times$ , we consider the action of  $P$  on the left of  $C_J(\sigma)$  defined by  $x \cdot a = x^{-1}ax^\sigma$  for all  $x \in P$  and all  $a \in C_J(\sigma)$ , and denote by  $\Omega_P(a)$  the orbit which contains an element  $a \in C_J(\sigma)$ . Then, for every  $x \in C_P(\sigma)$  the superclass of  $C_P(\sigma)$  which contains  $x$  can be defined to be the image  $\Phi(\Omega_P(a))$  where  $\Phi : C_J(\sigma) \rightarrow C_P(\sigma)$  is the Cayley transform and  $a \in C_J(\sigma)$  is such that  $x = \Phi(a)$ . On the other hand,  $P$  also acts on the left of the dual group  $C_J(\sigma)^\circ$  via the contragradient action given by  $x \cdot \lambda = x^{-1}\lambda x^\sigma$  for all  $x \in P$  and all  $\lambda \in C_J(\sigma)^\circ$ . For every  $\lambda \in C_J(\sigma)^\circ$ , we denote by  $\Omega_P(\lambda)$  the orbit which contains  $\lambda$ , and define the *supercharacter*  $\varsigma_\lambda$  of  $C_P(\sigma)$  to be the sum

$$\varsigma_\lambda = \sum_{\mu \in \Omega_P(\lambda)} \mu \circ \Psi$$

where  $\Psi: C_P(\sigma) \rightarrow C_{\mathcal{J}}(\sigma)$  is the inverse of the Cayley transform. We proved that for every  $\lambda \in C_{\mathcal{J}}(\sigma)^{\circ}$ , the function  $\varsigma_{\lambda}$  is in fact a character of  $C_P(\sigma)$  (Theorem 3.8), and that

$$\varsigma_{\lambda} = \sigma_{X(\lambda)} = \sum_{\chi \in X(\lambda)} \chi(1)\chi$$

where  $X(\lambda) = \text{Irr}_{\lambda}(C_P(\sigma))$  denotes the set of all irreducible constituents of  $\varsigma_{\lambda}$ . Also, we showed that as  $\lambda$  runs over a set of representatives for the orbits of  $P$  on  $C_{\mathcal{J}}(\sigma)^{\circ}$  the sets  $X(\lambda)$  partition  $\text{Irr}(C_P(\sigma))$ , and that together with the partition of  $C_P(\sigma)$  into superclasses they form a *supercharacter theory* for  $C_P(\sigma)$ ; notice that for every  $\lambda \in C_{\mathcal{J}}(\sigma)^{\circ}$ , the supercharacter  $\varsigma_{\lambda}$  is clearly constant on each superclass, and that the number of superclasses equals the number of supercharacters (Theorem 3.7).

## 6. THE CLASSICAL GROUPS

In this section, we illustrate our construction in the special case where  $\sigma: \mathcal{M}_n(\mathbb{k}) \rightarrow \mathcal{M}_n(\mathbb{k})$  is a canonical involution on  $\mathcal{M}_n(\mathbb{k})$  (as defined in the introduction); we will also assume that the upper unitriangular subgroup  $\text{UT}_n(\mathbb{k})$  of  $\text{GL}_n(\mathbb{k})$  is  $\sigma$ -invariant. Thus, if  $G = C_{\text{GL}_n(\mathbb{k})}(\sigma)$  denotes the  $\sigma$ -fixed subgroup of  $\text{GL}_n(\mathbb{k})$ , then  $G$  is one of the following (finite) classical groups of Lie type (defined over  $\mathbb{k}$ ): the *symplectic group*  $Sp_{2m}(\mathbb{k})$ , the *orthogonal groups*  $O_{2m}^+(\mathbb{k})$  or  $O_{2m+1}(\mathbb{k})$ , and the *unitary group*  $U_n(\mathbb{k})$ . (As we mentioned in the introduction, if  $\sigma$  is such that  $G$  is the orthogonal group  $O_{2m+2}^-(\mathbb{k})$ , then  $\text{UT}_n(\mathbb{k})$  has to be replaced by its maximal algebra subgroup  $\text{UT}_n(\mathbb{k}) \cap \text{UT}_n(\mathbb{k})^{\sigma}$ ; since the supercharacter theory of this subgroup has a slightly different parametrization than that of  $\text{UT}_n(F)$ , we skip the description and leave it as an exercise for the interested reader.) Thus, throughout the section,  $P$  will stand for the (upper) unitriangular group  $\text{UT}_n(\mathbb{k})$ , and we assume that the involution  $\sigma$  is chosen so that  $P$  is  $\sigma$ -invariant. It is straightforward to check that  $C_P(\sigma)$  consists of all (block) matrices of the form

$$(6a) \quad \begin{pmatrix} x & xu & xz \\ 0 & I_r & -\bar{u}^t J \\ 0 & 0 & J\bar{x}^{-t} J \end{pmatrix}$$

where  $J = J_m$  (see the introduction),  $x \in \text{UT}_m(\mathbb{k})$ ,  $u \in \mathcal{M}_{m \times r}(\mathbb{k})$  and  $z \in \mathcal{M}_m(\mathbb{k})$  satisfy the relations of the following table:

Classical group	Relations
$Sp_{2m}(\mathbb{k})$	$r = 0, Jz^t - zJ = 0,$
$O_{2m}^+(\mathbb{k})$	$r = 0, Jz^t + zJ = 0$
$O_{2m+1}(\mathbb{k})$	$r = 1, Jz^t + zJ = -uu^t$
$U_{2m}(\mathbb{k})$	$r = 0, J\bar{z}^t + zJ = 0$
$U_{2m+1}(\mathbb{k})$	$r = 1, J\bar{z}^t + zJ = -u\bar{u}^t$

We note that  $P = 1 + \mathcal{J}$  is the algebra group which is associated with the  $\sigma$ -invariant nilpotent uppertriangular subalgebra  $\mathcal{J} = \mathfrak{ut}_n(\mathbb{k})$  of  $\mathcal{M}_n(\mathbb{k})$ , and thus

$$C_P(\sigma) = \Phi(C_{\mathcal{J}}(\sigma))$$

where  $\Phi: \mathcal{J} \rightarrow P$  is the Cayley transform. Then,  $C_{\mathcal{J}}(\sigma)$  consists of all (block) matrices of the form

$$(6b) \quad \begin{pmatrix} a & u & w \\ 0 & 0_r & -\bar{u}^t J \\ 0 & 0 & -J\bar{a}^t J \end{pmatrix}$$

where where  $J = J_m$ ,  $x \in \text{UT}_m(\mathbb{k})$ ,  $u \in \mathcal{M}_{m \times r}(\mathbb{k})$  and  $z \in \mathcal{M}_m(\mathbb{k})$  satisfy the relations of the following table:

Classical group	Relations
$Sp_{2m}(\mathbb{k})$	$r = 0, Jw^t - wJ = 0,$
$O_{2m}^+(\mathbb{k})$	$r = 0, Jw^t + wJ = 0$
$O_{2m+1}(\mathbb{k})$	$r = 1, Jw^t + wJ = 0$
$U_{2m}(\mathbb{k})$	$r = 0, J\bar{w}^t + wJ = 0$
$U_{2m+1}(\mathbb{k})$	$r = 1, J\bar{w}^t + wJ = 0$

Superclasses and supercharacters of  $P$  are parametrised by pairs  $(\mathcal{D}, \varphi)$  where  $\mathcal{D}$  is a *basic subset* of  $[[n]] = \{(i, j): 1 \leq i < j \leq n\}$  and  $\varphi: \mathcal{D} \rightarrow \mathbb{k}^\times$  is any map. By definition, a subset  $\mathcal{D} \subseteq [[n]]$  is said to be *basic* if it contains at most one entry from each row and at most one root from each column; in other words,  $\mathcal{D}$  is basic if  $|\{j: i < j \leq n, (i, j) \in \mathcal{D}\}| \leq 1$  and

$|\{i: 1 \leq i < j, (i, j) \in \mathcal{D}\}| \leq 1$  for all  $1 \leq i, j \leq n$ . Henceforth, we will refer to such a pair  $(\mathcal{D}, \varphi)$  as a *basic pair* for  $P$ . For  $(i, j) \in [[n]]$ , we denote by  $e_{i,j}$  the  $(i, j)$ th unit matrix with 1 in the  $(i, j)$ th position and 0's elsewhere; hence,  $\{e_{i,j}: (i, j) \in [[n]]\}$  is the standard  $\mathbb{k}$ -basis of  $\mathcal{J}$ . For every basic pair  $(\mathcal{D}, \varphi)$ , we define

$$e_{\mathcal{D}}(\varphi) = \sum_{(i,j) \in \mathcal{D}} \varphi(i, j) e_{i,j} \in \mathcal{J};$$

notice that, if  $\mathcal{D}$  is empty, then the sum is empty, and hence  $e_{\mathcal{D}}(\varphi) = 0$  (by convention, in this extreme case, we agree that  $\varphi$  is the empty function). In virtue of Lemma 2.1, we define the *superclass*  $\widehat{\mathcal{K}}_{\mathcal{D}}(\varphi)$  of  $P$  to be the subset

$$\widehat{\mathcal{K}}_{\mathcal{D}}(\varphi) = \Phi(P e_{\mathcal{D}}(\varphi) P)$$

of  $P$ ; notice that  $\widehat{\mathcal{K}}_{\mathcal{D}}(\varphi)$  contains the element  $\Phi(e_{\mathcal{D}}(\varphi)) = 1 + 2e_{\mathcal{D}}(\varphi)$ . We have:

- (1) If  $\widehat{\mathcal{K}}$  is a superclass of  $P$ , then  $\widehat{\mathcal{K}} = \widehat{\mathcal{K}}_{\mathcal{D}}(\varphi)$  for some basic pair  $(\mathcal{D}, \varphi)$ .
- (2) If  $(\mathcal{D}, \varphi)$  and  $(\mathcal{D}', \varphi')$  are basic pairs for  $P$ , then  $\widehat{\mathcal{K}}_{\mathcal{D}}(\varphi) \cap \widehat{\mathcal{K}}_{\mathcal{D}'}(\varphi') \neq \emptyset$  if and only if  $(\mathcal{D}, \varphi) = (\mathcal{D}', \varphi')$ .

As in Section 2, the superclasses of the  $\sigma$ -fixed subgroup  $C_P(\sigma)$  are defined to be the non-empty intersections

$$\mathcal{K}_{\mathcal{D}}(\varphi) = \widehat{\mathcal{K}}_{\mathcal{D}}(\varphi) \cap C_P(\sigma)$$

where  $(\mathcal{D}, \varphi)$  is a basic pair for  $P$ ; moreover, by Proposition 2.3, this intersection is non-empty if and only if  $\widehat{\mathcal{K}}_{\mathcal{D}}(\varphi)$  is  $\sigma$ -invariant. In fact, for a fixed basic pair  $(\mathcal{D}, \varphi)$ , the action of  $\sigma$  defines a superclass  $\widehat{\mathcal{K}}_{\mathcal{D}}(\varphi)^{\sigma}$  (by Lemma 2.1), and thus there is a basic pair  $(\mathcal{D}^{\sigma}, \varphi^{\sigma})$  such that

$$\widehat{\mathcal{K}}_{\mathcal{D}}(\varphi)^{\sigma} = \widehat{\mathcal{K}}_{\mathcal{D}^{\sigma}}(\varphi^{\sigma}) = \Phi(P e_{\mathcal{D}^{\sigma}}(\varphi^{\sigma}) P);$$

in particular, it follows that  $\widehat{\mathcal{K}}_{\mathcal{D}}(\varphi)$  is  $\sigma$ -invariant if and only if  $\mathcal{D}^{\sigma} = \mathcal{D}$  and  $\varphi^{\sigma} = \varphi$ . By Lemma 2.2, we know that

$$\widehat{\mathcal{K}}_{\mathcal{D}^{\sigma}}(\varphi^{\sigma}) = \Phi(P e_{\mathcal{D}}(\varphi)^{\sigma} P).$$

Since  $\sigma$  is canonical, we have  $(e_{i,j})^{\sigma} = -\sigma(e_{i,j}) = \pm e_{n-j+1, n-i+1}$  for all  $(i, j) \in [[n]]$ , and so  $e_{\mathcal{D}}(\varphi)^{\sigma} = e_{\mathcal{D}^{\sigma}}(\varphi^{\sigma})$ . In particular, we conclude that  $\widehat{\mathcal{K}}_{\mathcal{D}}(\varphi)$  is  $\sigma$ -invariant if and only if  $e_{\mathcal{D}}(\varphi) \in C_{\mathcal{J}}(\sigma)$ , and thus Theorem 2.4 implies the following result. Here, and henceforth, we say that a basic pair  $(\mathcal{D}, \varphi)$  for  $P$  is  $\sigma$ -*invariant* if  $(\mathcal{D}^{\sigma}, \varphi^{\sigma}) = (\mathcal{D}, \varphi)$  (hence,  $(\mathcal{D}, \varphi)$  is  $\sigma$ -invariant if and only if  $e_{\mathcal{D}}(\varphi) \in C_{\mathcal{J}}(\sigma)$ ); similarly, we say that a basic subset  $\mathcal{D}$  of  $[[n]]$  is  $\sigma$ -*invariant* if  $\mathcal{D}^{\sigma} = \mathcal{D}$ .

**Proposition 6.1.** *There is a one-to-one correspondence between superclasses of  $C_P(\sigma)$  and  $\sigma$ -invariant basic pairs for  $P$ , where the superclass  $\mathcal{K}_{\mathcal{D}}(\varphi)$  which corresponds to a  $\sigma$ -invariant*

basic pair  $(\mathcal{D}, \varphi)$  is given by

$$\mathcal{K}_{\mathcal{D}}(\varphi) = \{\Phi(xe_{\mathcal{D}}(\varphi)x^{-\sigma}) : x \in P\}.$$

*Remark 6.2.* It is clear that every  $\sigma$ -invariant basic subset  $\mathcal{D}$  of  $[[n]]$  decomposes as a disjoint union  $\mathcal{D} = \mathcal{D}_1 \sqcup (\mathcal{D}_1)^{\sigma} \sqcup \mathcal{D}_0$  where

$$\mathcal{D}_1 = \{(i, j) \in \mathcal{D} : i \leq m, j < n - i + 1\}, \text{ and}$$

$$\mathcal{D}_0 = \{(i, n - i + 1) : i \leq m, (i, n - i + 1) \in \mathcal{D}\};$$

notice that  $(\mathcal{D}_1)^{\sigma} = \{(n - j + 1, n - i + 1) : (i, j) \in \mathcal{D}_1\}$  and that  $(\mathcal{D}_0)^{\sigma} = \mathcal{D}_0$ . On the other hand, if  $\varphi : \mathcal{D} \rightarrow \mathbb{k}^{\times}$  is any map and  $|\mathbb{k}^{\sigma}| = q$ , then  $e_{\mathcal{D}}(\varphi) \in C_{\mathcal{J}}(\sigma)$  if and only if

$$(6c) \quad \varphi(n - j + 1, n - i + 1) = \begin{cases} -\varphi(i, j)^q, & \text{if } j \leq m + r, \\ -\varphi(i, j)^q, & \text{if } m + r < j \text{ and } G \neq Sp_{2m}(\mathbb{k}), \\ \varphi(i, j), & \text{if } m < j \text{ and } G = Sp_{2m}(\mathbb{k}), \end{cases}$$

for all  $(i, j) \in \mathcal{D}_1 \cup \mathcal{D}_0$ . In particular, we deduce that

- if, either  $G = O_{2m}^{+}(\mathbb{k})$ , or  $G = O_{2m+1}(\mathbb{k})$ , then  $\varphi(i, n - i + 1) = 0$  for all  $1 \leq i \leq m$ ;
- if  $G = U_n(\mathbb{k})$ , then  $\varphi(i, n - i + 1) + \varphi(i, n - i + 1)^q = 0$  for all  $1 \leq i \leq m$ .

Next, we consider supercharacters, and we start by recalling the construction of the supercharacter  $\widehat{\xi}_{\mathcal{D}}(\varphi)$  of  $P$  which is associated with a given basic pair  $(\mathcal{D}, \varphi)$ ; for the details, we refer to Section 3. We fix any non-trivial  $\sigma$ -invariant linear character  $\vartheta : \mathbb{k}^+ \rightarrow \mathbb{C}^{\times}$ ; thus, since  $\sigma$  acts on  $\mathbb{k}$  as the Frobenius automorphism, we have  $\vartheta(\alpha^q) = \vartheta(\alpha)$  for all  $\alpha \in \mathbb{k}$ . Then, we define the linear character  $\lambda_{\mathcal{D}}(\varphi) : \mathcal{J}^+ \rightarrow \mathbb{C}^{\times}$  of the additive group  $\mathcal{J}^+$  by the rule

$$\lambda_{\mathcal{D}}(\varphi)(a) = \prod_{(i,j) \in \mathcal{D}} \vartheta(\varphi(i, j)a_{ij})$$

for all  $a \in \mathcal{J}$ , and let

$$L_{\mathcal{D}} = L(\lambda_{\mathcal{D}}(\varphi)) = \{x \in P : x\lambda_{\mathcal{D}}(\varphi) = \lambda_{\mathcal{D}}(\varphi)\}$$

be the centraliser of  $\lambda_{\mathcal{D}}(\varphi)$  with respect to the left  $P$ -action on  $\mathcal{J}$ . It is routine to check that  $L_{\mathcal{D}}$  consists of all matrices  $x \in P$  which satisfy  $x_{ik} = 0$  for all  $(i, j) \in \mathcal{D}$  and all  $i < k < j$  (hence,  $L_{\mathcal{D}}$  does not depend on the map  $\varphi$ ), and that the mapping

$$x \mapsto \lambda_{\mathcal{D}}(\varphi)(\Psi(x))$$

defines a linear character  $\widehat{\vartheta}_{\mathcal{D}}(\varphi) : L_{\mathcal{D}} \rightarrow \mathbb{C}^{\times}$ . Then, we define the *supercharacter*  $\widehat{\xi}_{\mathcal{D}}(\varphi)$  of  $P$  to be the induced character

$$\widehat{\xi}_{\mathcal{D}}(\varphi) = \widehat{\vartheta}_{\mathcal{D}}(\varphi)^P.$$

In particular, if  $\mathcal{D} = \{(i, j)\}$  consists of a single entry  $(i, j) \in [[n]]$  and  $\varphi: \mathcal{D} \rightarrow \mathbb{k}^\times$  is given by  $\varphi(i, j) = \alpha \in \mathbb{k}^\times$ , then we write  $\lambda_{i,j}(\alpha)$ ,  $\widehat{\vartheta}_{i,j}(\alpha)$  and  $\widehat{\xi}_{i,j}(\alpha)$  instead of  $\lambda_{\mathcal{D}}(\varphi)$ ,  $\widehat{\vartheta}_{\mathcal{D}}(\varphi)$  and  $\widehat{\xi}_{\mathcal{D}}(\varphi)$ , respectively; if this is the case, then we refer to the supercharacter  $\widehat{\xi}_{i,j}(\alpha) = \widehat{\vartheta}_{i,j}(\alpha)^P$  as the  $(i, j)$ th elementary character of  $P$  associated with  $\alpha$ . In the general case, since  $L_{\mathcal{D}} = \bigcap_{(i,j) \in \mathcal{D}} L_{i,j}$  where we write  $L_{i,j} = L_{\{(i,j)\}}$ , it is not difficult to prove that the supercharacter  $\widehat{\xi}_{\mathcal{D}}(\varphi)$  factorises as the product

$$(6d) \quad \widehat{\xi}_{\mathcal{D}}(\varphi) = \prod_{(i,j) \in \mathcal{D}} \widehat{\xi}_{i,j}(\varphi(i, j))$$

of elementary supercharacters (see, for example, [6, Theorem 1]); we also note that every elementary supercharacter is in fact an irreducible character of  $P$  (see [4, Lemma 2], or [16, Corollary 5.11]).

If  $(\mathcal{D}, \varphi)$  is any basic pair for  $P$ , then the action of  $\sigma$  defines a supercharacter  $\widehat{\xi}_{\mathcal{D}}(\varphi)^\sigma$  which corresponds to the linear character  $\lambda_{\mathcal{D}}(\varphi)^\sigma$  of  $\mathcal{J}^+$  (by Lemma 3.2). Since  $\vartheta$  is  $\sigma$ -invariant, it is easy to check that for all  $(i, j) \in [[n]]$  and all  $\alpha \in \mathbb{k}^\times$  we have

$$\lambda_{i,j}(\alpha) = \begin{cases} \lambda_{n-j+1, n-i+1}(-\alpha^q), & \text{if } j \leq m+r, \\ \lambda_{n-j+1, n-i+1}(-\alpha^q), & \text{if } m+r < j \text{ and } G \neq Sp_{2m}(\mathbb{k}), \\ \lambda_{n-j+1, n-i+1}(\alpha), & \text{if } m < j \text{ and } G = Sp_{2m}(\mathbb{k}), \end{cases}$$

where  $q = |\mathbb{k}^\sigma|$ , and this clearly implies that  $\lambda_{\mathcal{D}}(\varphi)^\sigma = \lambda_{\mathcal{D}^\sigma}(\varphi^\sigma)$  where the basic pair  $(\mathcal{D}^\sigma, \varphi^\sigma)$  is as above. Therefore, we have

$$(6e) \quad \widehat{\xi}_{\mathcal{D}}(\varphi)^\sigma = \widehat{\xi}_{\mathcal{D}^\sigma}(\varphi^\sigma),$$

and it follows that  $\widehat{\xi}_{\mathcal{D}}(\varphi)$  is  $\sigma$ -invariant if and only if the basic pair  $(\mathcal{D}, \varphi)$  is  $\sigma$ -invariant. By Proposition 3.3, we obtain the following result.

**Proposition 6.3.** *There is a one-to-one correspondence between supercharacters of  $C_P(\sigma)$  and  $\sigma$ -invariant basic pairs for  $P$ .*

In what follows, we fix an arbitrary  $\sigma$ -invariant basic pair  $(\mathcal{D}, \varphi)$  for  $P$ , and consider the supercharacter of  $C_P(\sigma)$  which is associated with  $(\mathcal{D}, \varphi)$ . On the one hand, let

$$\Omega_{\mathcal{D}}(\varphi) = \{x^{-1}\lambda_{\mathcal{D}}(\varphi)x^\sigma : x \in P\}$$

be the subset of  $P\lambda_{\mathcal{D}}(\varphi)P$  consisting of  $\sigma$ -fixed elements (see Proposition 3.4), and define the map  $\varsigma_{\mathcal{D}}(\varphi): C_P(\sigma) \rightarrow \mathbb{C}$  by the rule

$$(6f) \quad \varsigma_{\mathcal{D}}(\varphi)(x) = \sum_{\lambda \in \Omega_{\mathcal{D}}(\varphi)} \mu(\Psi(x))$$

for all  $x \in C_P(\sigma)$ . By Theorem 3.8,  $\varsigma_{\mathcal{D}}(\varphi)$  is a character of  $C_P(\sigma)$ , and in fact

$$\varsigma_{\mathcal{D}}(\varphi) = \sum_{\chi \in X_{\mathcal{D}}(\varphi)} \chi(1) \chi$$

where  $X_{\mathcal{D}}(\varphi) = X(\lambda_{\mathcal{D}}(\varphi))$  denotes the set consisting of all irreducible constituents of  $\varsigma_{\mathcal{D}}(\varphi)$ ; we recall that  $X_{\mathcal{D}}(\varphi)$  can also be described as the set consisting of all irreducible characters  $\chi \in \text{Irr}(C_P(\sigma))$  such that the Glauberman correspondent  $\widehat{\chi} \in \text{Irr}_{\sigma}(P)$  of  $\chi$  is a constituent of the supercharacter  $\widehat{\xi}_{\mathcal{D}}(\varphi)$  of  $P$ . The results of Section 3 imply the following.

**Theorem 6.4.** *If  $\mathfrak{D}$  denotes the set of all  $\sigma$ -invariant basic pairs for  $P$ , then the sets  $\mathcal{X} = \{\varsigma_{\mathcal{D}}(\varphi) : (\mathcal{D}, \varphi) \in \mathfrak{D}\}$  and  $\mathcal{Y} = \{\mathcal{K}_{\mathcal{D}}(\varphi) : (\mathcal{D}, \varphi) \in \mathfrak{D}\}$  form a supercharacter theory for  $C_P(\sigma)$ .*

Although the supercharacters are defined in a different way, in the case of the symplectic and orthogonal groups this supercharacter theory for  $C_P(\sigma)$  turns out to be the same as the one described in the papers [7, 8, 9]; in fact, [9, Theorem 6.1] asserts that, up to the multiplication by a positive integer, the supercharacter  $\varsigma_{\mathcal{D}}(\varphi)$  can be obtained by inducting a linear character of a suitable subgroup of  $C_P(\sigma)$ . To see this, we first define the subgroup  $Q_{\mathcal{D}}$  of  $P$  as follows: for every  $(i, j) \in [[n]]$  let

$$\begin{aligned} Q_{i,j} &= L_{i,j}, \quad \text{if } j \leq m, \\ Q_{i,j} &= \{x \in P : x_{i,k} = x_{k,j} = 0 \text{ for all } i < k \leq m\}, \quad \text{if } i \leq m < j; \\ Q_{i,j} &= (L_{n-j+1, n-i+1})^{\sigma}, \quad \text{if } m < i; \end{aligned}$$

then,

$$Q_{\mathcal{D}} = \bigcap_{(i,j) \in \mathcal{D}} Q_{i,j}.$$

On the other hand, for every map  $\varphi : \mathcal{D} \rightarrow \mathbb{k}^{\times}$ , we define  $\widehat{\tau}_{\mathcal{D}}(\varphi) : Q_{\mathcal{D}} \rightarrow \mathbb{C}^{\times}$  by

$$\widehat{\tau}_{\mathcal{D}}(\varphi)(x) = \lambda_{\mathcal{D}}(\varphi)(\Psi(x))$$

for all  $x \in Q_{\mathcal{D}}$ . It is easy to check that  $\widehat{\tau}_{\mathcal{D}}(\varphi)$  is a linear character of  $Q_{\mathcal{D}}$ ; moreover, by [8, Lemma 2.1] it follows that

$$(6g) \quad \widehat{\xi}_{\mathcal{D}}(\varphi) = \widehat{\tau}_{\mathcal{D}}(\varphi)^P$$

(see also Proposition 3.1). If the basic pair  $(\mathcal{D}, \varphi)$  is  $\sigma$ -invariant, then it is straightforward to check that the subgroup  $Q_{\mathcal{D}}$  and the linear character  $\widehat{\tau}_{\mathcal{D}}(\varphi)$  are both  $\sigma$ -invariant; if this is the case, we denote by  $\tau_{\mathcal{D}}(\varphi)$  the restriction of  $\widehat{\tau}_{\mathcal{D}}(\varphi)$  to the  $\sigma$ -fixed subgroup  $C_{Q_{\mathcal{D}}}(\sigma)$ , and define

$$(6h) \quad \xi_{\mathcal{D}}(\varphi) = \tau_{\mathcal{D}}(\varphi)^{C_P(\sigma)}.$$

We claim that there exists a positive integer  $n_{\mathcal{D}, \varphi}$  such that  $\varsigma_{\mathcal{D}}(\varphi) = n_{\mathcal{D}, \varphi} \xi_{\mathcal{D}}(\varphi)$ . To see this, we first prove the following general result (which extends Lemma 4.2).

**Proposition 6.5.** *Let  $\mathcal{J}$  be a  $\sigma$ -invariant nilpotent subalgebra of  $\mathcal{A}$ , and let  $P = 1 + \mathcal{J}$ . Let  $\mathcal{I}$  be a  $\sigma$ -invariant subalgebra of  $\mathcal{J}$ , let  $Q = 1 + \mathcal{I}$ , let  $\widehat{\vartheta} \in \text{Irr}_\sigma(Q)$  and let  $\vartheta = \pi_Q(\widehat{\vartheta}) \in \text{Irr}(C_Q(\sigma))$  be the Glauberman correspondent of  $\widehat{\vartheta}$ . Moreover, let  $\chi \in \text{Irr}(C_P(\sigma))$ , and let  $\widehat{\chi} \in \text{Irr}_\sigma(P)$  be such that  $\pi_P(\widehat{\chi}) = \chi$ . Then,  $\langle \chi, \vartheta^{C_P(\sigma)} \rangle \neq 0$  if and only if  $\langle \widehat{\chi}, \widehat{\vartheta}^P \rangle \neq 0$ .*

*Proof.* We proceed by induction on  $\dim \mathcal{J}$ . Firstly, suppose that  $\mathcal{I} + \mathcal{J}^2 = \mathcal{J}$ . Then, by [21, Lemma 3.1], we have  $\mathcal{I} = \mathcal{J}$ ; hence,  $Q = P$  and there is nothing to prove. Otherwise, let  $N = 1 + (\mathcal{I} + \mathcal{J}^2)$ ; hence,  $Q \subseteq N \subsetneq P$ . Then, since  $\mathcal{J}^2 \subseteq \mathcal{I} + \mathcal{J}^2$  and since  $\mathcal{J}^2$  is clearly  $\sigma$ -invariant,  $N$  is a  $\sigma$ -invariant normal subgroup of  $P$ . Now, let us assume that  $\langle \chi, \vartheta^{C_P(\sigma)} \rangle \neq 0$ . Then, by Frobenius reciprocity, we have  $\langle \chi, (\vartheta^{C_N(\sigma)})^{C_P(\sigma)} \rangle = \langle \chi_{C_N(\sigma)}, \vartheta^{C_N(\sigma)} \rangle$ , and thus there exists  $\tau \in \text{Irr}(C_N(\sigma))$  such that  $\langle \tau, \chi_{C_N(\sigma)} \rangle \neq 0$  and  $\langle \tau, \vartheta^{C_N(\sigma)} \rangle \neq 0$ . Since  $\langle \tau, \chi_{C_N(\sigma)} \rangle = \langle \tau^{C_P(\sigma)}, \chi \rangle$ , [20, Theorem (13.29)] implies that  $\langle \widehat{\tau}^P, \widehat{\chi} \rangle \neq 0$  where  $\widehat{\tau} \in \text{Irr}_\sigma(N)$  is such that  $\pi_N(\widehat{\tau}) = \tau$ . On the other hand, by induction, we also have  $\langle \widehat{\tau}, \widehat{\vartheta}^N \rangle \neq 0$ , and thus  $\widehat{\tau}^P$  is a constituent of  $\widehat{\vartheta}^P = (\widehat{\vartheta}^N)^P$ . Since  $\widehat{\chi}$  is a constituent of  $\widehat{\tau}^P$ , we conclude that  $\langle \widehat{\chi}, \widehat{\vartheta}^P \rangle \neq 0$ , as required. Conversely, suppose that  $\langle \widehat{\chi}, \widehat{\vartheta}^P \rangle \neq 0$ ; thus,  $\langle \widehat{\chi}_N, \widehat{\vartheta}^N \rangle \neq 0$  (by Frobenius reciprocity). By [20, Theorem (13.27)], there exists  $\widehat{\tau} \in \text{Irr}_\sigma(N)$  such that  $\langle \widehat{\tau}, \widehat{\chi}_N \rangle \neq 0$ . Then,  $\langle \widehat{\tau}, \widehat{\vartheta}^N \rangle \neq 0$ , and so by induction we obtain  $\langle \tau, \vartheta^{C_N(\sigma)} \rangle \neq 0$  where  $\tau = \pi_N(\widehat{\tau}) \in \text{Irr}(C_N(\sigma))$ . Since  $\langle \widehat{\tau}, \widehat{\chi}_N \rangle = \langle \widehat{\tau}^P, \widehat{\chi} \rangle$ , [20, Theorem (13.29)] implies that  $\langle \tau^{C_P(\sigma)}, \chi \rangle \neq 0$ . Since  $\tau^{C_P(\sigma)}$  is a constituent of  $\vartheta^{C_P(\sigma)} = (\vartheta^{C_N(\sigma)})^{C_P(\sigma)}$ , we conclude that  $\langle \chi, \vartheta^{C_P(\sigma)} \rangle \neq 0$ , and this completes the proof.  $\square$

We are now able to prove the following result.

**Lemma 6.6.** *If  $(\mathcal{D}, \varphi)$  be a  $\sigma$ -invariant basic pair for  $P$ , then the characters  $\xi_{\mathcal{D}}(\varphi)$  and  $\varsigma_{\mathcal{D}}(\varphi)$  of  $C_P(\sigma)$  have the same irreducible constituents. In particular, if  $(\mathcal{D}, \varphi)$  and  $(\mathcal{D}', \varphi')$  are  $\sigma$ -invariant basic pairs for  $P$ , then  $\langle \xi_{\mathcal{D}}(\varphi), \xi_{\mathcal{D}'}(\varphi') \rangle \neq 0$  if and only if  $(\mathcal{D}, \varphi) = (\mathcal{D}', \varphi')$ .*

*Proof.* If  $\chi \in \text{Irr}(C_P(\sigma))$  is an irreducible constituent of  $\xi_{\mathcal{D}}(\varphi) = \tau_{\mathcal{D}}(\varphi)^{C_P(\sigma)}$ , then the previous proposition asserts that the Glauberman correspondent  $\widehat{\chi} \in \text{Irr}_\sigma(P)$  of  $\chi$  is a constituent of  $\widehat{\xi}_{\mathcal{D}}(\varphi) = \widehat{\tau}_{\mathcal{D}}(\varphi)^{C_P(\sigma)}$ , and thus  $\chi$  is an irreducible constituent of  $\varsigma_{\mathcal{D}}(\varphi)$  (by Theorem 3.8). Conversely, if  $\chi \in \text{Irr}(C_P(\sigma))$  is an irreducible constituent of  $\varsigma_{\mathcal{D}}(\varphi)$ , then  $\widehat{\chi}$  is an irreducible constituent of  $\widehat{\xi}_{\mathcal{D}}(\varphi)$ , and so  $\chi$  is an irreducible constituent of  $\xi_{\mathcal{D}}(\varphi)$  (by the previous proposition). For the last assertion, it is enough to recall that  $\langle \varsigma_{\mathcal{D}}(\varphi), \varsigma_{\mathcal{D}'}(\varphi') \rangle \neq 0$  if and only if  $(\mathcal{D}, \varphi) = (\mathcal{D}', \varphi')$ .  $\square$

We next show that  $\xi_{\mathcal{D}}(\varphi)$  is a superclass function on  $C_P(\sigma)$ . Since the basic subset  $\mathcal{D} \subseteq [[n]]$  is  $\sigma$ -invariant, we have a decomposition  $\mathcal{D} = \mathcal{D}_1 \sqcup (\mathcal{D}_1)^\sigma \sqcup \mathcal{D}_0$  where  $\mathcal{D}_1$  and  $\mathcal{D}_0$  are as in Remark 6.2. On the other hand, since the basic pair  $(\mathcal{D}, \varphi)$  is  $\sigma$ -invariant, Eqs. (6d) and (6e) imply that the supercharacter  $\widehat{\xi}_{\mathcal{D}}(\varphi)$  factorises as the product

$$\widehat{\xi}_{\mathcal{D}}(\varphi) = \widehat{\xi}_{\mathcal{D}_1}(\varphi_1) \widehat{\xi}_{\mathcal{D}_1}(\varphi_1)^\sigma \widehat{\xi}_{\mathcal{D}_0}(\varphi_0)$$

where  $\varphi_1$  and  $\varphi_0$  denote the restriction of  $\varphi$  to  $\mathcal{D}_1$  and  $\mathcal{D}_0$ , respectively. Since  $\widehat{\xi}_{\mathcal{D}_1}(\varphi_1)$  and  $\widehat{\xi}_{\mathcal{D}_1}(\varphi_1)^\sigma$  have the same restriction to  $C_P(\sigma)$ , we conclude that

$$\begin{aligned}\widehat{\xi}_{\mathcal{D}}(\varphi)_{C_P(\sigma)} &= (\widehat{\xi}_{\mathcal{D}_1}(\varphi_1)_{C_P(\sigma)})^2 \cdot \widehat{\xi}_{\mathcal{D}_0}(\varphi_0)_{C_P(\sigma)} \\ &= \prod_{(i,j) \in \mathcal{D}_1} (\widehat{\xi}_{i,j}(\varphi(i,j))_{C_P(\sigma)})^2 \cdot \prod_{(i,j) \in \mathcal{D}_0} \widehat{\xi}_{i,j}(\varphi(i,j))_{C_P(\sigma)}.\end{aligned}$$

*Remark 6.7.* We observe that, for all  $(i,j) \in [[n]]$  and all  $\alpha \in \mathbb{k}^\times$ , the square power  $\widehat{\xi}_{i,j}(\alpha)^2$  is a superclass function of  $P$ , and thus it decomposes as a linear combination of supercharacters (with integer coefficients); furthermore, from [2, Lemma 11] (see also Proposition 3.1) it follows that  $\widehat{\xi}_{i,j}(2\alpha)$  is an irreducible constituent of  $\widehat{\xi}_{i,j}(\alpha)^2$  with multiplicity equal to  $1 + (q-1)(j-i+1)$  where  $q = |\mathbb{k}^\sigma|$ .

Henceforth, for every  $(i,j) \in [[n]]$  with  $j \leq n - i + 1$  and every  $\alpha \in \mathbb{k}^\times$ , we will simplify the notation and write  $\xi_{i,j}(\alpha)$  (resp.,  $\varsigma_{i,j}(\alpha)$ ) to denote the character  $\xi_{\mathcal{D}}(\varphi)$  (resp., the supercharacter  $\varsigma_{\mathcal{D}}(\varphi)$ ) of  $C_P(\sigma)$  where  $(\mathcal{D}, \varphi)$  is the  $\sigma$ -invariant basic pair with  $\mathcal{D} = \{(i,j), (n-j+1, n-i+1)\}$  and  $\alpha = \varphi(i,j)$ ; as before, we refer to  $\xi_{i,j}(\alpha)$  as the  $(i,j)$ th *elementary character* of  $C_P(\sigma)$  associated with  $\alpha$ . Similarly to the case of the unitriangular group, we have the following factorisation; for a proof, see [7, Proposition 3].

**Theorem 6.8.** *If  $(\mathcal{D}, \varphi)$  is a  $\sigma$ -invariant basic pair for  $P$ , then*

$$\xi_{\mathcal{D}}(\varphi) = \prod_{(i,j) \in \mathcal{D}'} \xi_{i,j}(\varphi(i,j))$$

where  $\mathcal{D}' = \{(i,j) \in \mathcal{D} : j \leq n - i + 1\}$ .

In view of this theorem, the goal of proving that the  $\xi_{\mathcal{D}}(\varphi)$  is a superclass function of  $C_P(\sigma)$  reduces to proving that this holds for every elementary character.

**Lemma 6.9.** *Let  $(i,j) \in [[n]]$  be such that  $j \leq n - i + 1$ , and let  $\alpha \in \mathbb{k}^\times$ . Then,  $\xi_{i,j}(\alpha) = \widehat{\xi}_{i,j}(2\alpha)_{C_P(\sigma)}$ , and hence  $\xi_{i,j}(\alpha)$  is a superclass function on  $C_P(\sigma)$ . In particular, there exists a constant  $n_{i,j}(\alpha)$  such that  $\xi_{i,j}(\alpha) = n_{i,j}(\alpha) \varsigma_{i,j}(\alpha)$ .*

*Proof.* For simplicity, we set  $\widehat{\xi} = \widehat{\xi}_{i,j}(2\alpha)$ ; as for Eq. (6g), [8, Lemma 2.1] implies that  $\widehat{\xi} = \widehat{\tau}^P$  where  $\widehat{\tau} = \widehat{\tau}_{i,j}(\alpha)$  is the linear character of  $Q = Q_{i,j}$  defined by

$$\widehat{\tau}(x) = \vartheta(2\alpha x_{i,j})$$

for all  $x \in Q$ . Since  $P = QC_P(\sigma)$ , we obtain

$$\widehat{\xi}_{C_P(\sigma)} = (\widehat{\tau}_{Q \cap C_P(\sigma)})^{C_P(\sigma)} = (\widehat{\tau}_{C_Q(\sigma)})^{C_P(\sigma)}$$

(by Mackey's criterion; see [20, Exercise 6.1]). Since  $\widehat{\tau}(x) = \vartheta(2\alpha x_{i,j}) = \vartheta(\alpha x_{i,j})^2 = \tau_{\mathcal{D}}(\varphi)(x)$  for all  $x \in C_Q(\sigma)$ , we conclude that  $\widehat{\xi}_{C_P(\sigma)} = \xi_{i,j}(\alpha)$ , and thus  $\xi_{i,j}(\alpha)$  is a superclass function on

$C_P(\sigma)$  (because  $\widehat{\xi}$  is a superclass function on  $P$ ). It follows that  $\xi_{i,j}(\alpha)$  is a linear combination of the supercharacters of  $C_P(\sigma)$ , and hence  $\xi_{i,j}(\alpha)$  must be a multiple of  $\varsigma_{i,j}(\alpha)$  (by Lemma 6.6).  $\square$

On the other hand, we consider the restriction to  $C_P(\sigma)$  of a  $\sigma$ -invariant elementary character  $\widehat{\xi}_{i,n-i+1}(\alpha)$  where  $i \leq m$  and  $\alpha \in \mathbb{k}^\times$ ; the assumption of being  $\sigma$ -invariant implies that, either  $G = Sp_{2m}(\mathbb{k})$ , or  $G = U_n(\mathbb{k})$  and  $\alpha \in \mathbb{k}$  satisfies  $\alpha^q = -\alpha$  where  $q = |\mathbb{k}^\sigma|$ . Since  $\widehat{\xi}_{i,n-i+1}(\alpha)$  is an irreducible character of  $P$  ([4, Lemma 2], or [16, Corollary 5.11]), Glauberman's Theorem guarantees that its restriction to  $C_P(\sigma)$  has a unique irreducible constituent with odd multiplicity, and this clearly implies that there exists a positive integer  $m$  such that  $\xi_{i,n-i+1}(\alpha) = m\chi$  where  $\chi = \pi_P(\widehat{\xi}_{i,n-i+1}(\alpha))$ . In fact, we have the following.

**Lemma 6.10.** *If  $i < m$  and  $\alpha \in \mathbb{k}^\times$  are as above, then  $\xi_{i,n-i+1}(\alpha)$  is an irreducible character of  $C_P(\sigma)$ , and  $\varsigma_{i,n-i+1}(\alpha) = q^{m-i+1}\xi_{i,n-i+1}(\alpha)$  where  $q = |\mathbb{k}^\sigma|$ .*

*Proof.* For simplicity, we set  $\xi = \xi_{i,n-i+1}(\alpha)$  and  $\tau = \tau_{i,n-i+1}(\alpha)$ ; hence,  $\tau$  is a linear character of  $Q = Q_{i,n-i+1}$  and  $\xi = \tau^{C_P(\sigma)}$ . We observe that the group  $C_P(\sigma)$  factorises as the semidirect product

$$C_P(\sigma) = P_0 \ltimes N$$

where  $P_0$  is a subgroup (naturally) isomorphic to the unitriangular group  $UT_m(q)$  and  $N$  is a normal subgroup of nilpotency class less than or equal 2; referring to Eq. (6a),  $P_0$  consists of all (block) matrices with  $u = 0$  and  $z = 0$ , and  $N$  consists of all matrices with  $x = I_m$ . It is routine to check that  $Q$  equals the inertia group  $I_P(\tau_N)$  in  $P$  of the restriction  $\tau_N$  of  $\tau$  to  $N$ ; in other words, this means that  $Q = \{x \in P : \tau(xyx^{-1}) = \tau(y) \text{ for all } y \in N\}$ . By Clifford's theorem (see [21, Theorem 6.11]), we conclude that  $\xi = \tau^P$  is an irreducible character. By the above, this implies that  $\xi = \pi_P(\widehat{\xi}_{i,n-i+1}(\alpha))$ , and thus  $\varsigma_{i,n-i+1}(\alpha) = \xi(1)\xi$  (by Theorem 3.8). The result follows because  $\xi(1) = |C_P(\sigma) : C_Q(\sigma)| = q^{m-i+1}$ .  $\square$

Finally, we deduce the following (required) result.

**Proposition 6.11.** *If  $(\mathcal{D}, \varphi)$  is a  $\sigma$ -invariant basic pair for  $P$ , then  $\xi_{\mathcal{D}}(\varphi)$  is a superclass function of  $C_P(\sigma)$ , and hence there exists a constant  $n_{\mathcal{D}, \varphi}$  such that  $\xi_{\mathcal{D}}(\varphi) = n_{\mathcal{D}, \varphi}\varsigma_{\mathcal{D}}(\varphi)$ .*

*Proof.* By Theorem 6.8 and by the two previous lemmas, it follows that  $\xi_{\mathcal{D}}(\varphi)$  is in fact a superclass function. Since supercharacters form a basis of the complex vector space consisting of all superclass functions (because they are orthogonal and in the same number as superclasses), we conclude that  $\xi_{\mathcal{D}}(\varphi)$  is a linear combination of supercharacters, and Lemma 6.6 implies that  $\xi_{\mathcal{D}}(\varphi)$  must be a multiple of  $\varsigma_{\mathcal{D}}(\varphi)$ .  $\square$

As a consequence, we obtain the following result (see [9] for the symplectic and orthogonal cases).

**Theorem 6.12.** *If  $\mathfrak{D}$  denotes the set of all  $\sigma$ -invariant basic pairs for  $P$ , then the sets  $\mathcal{X}' = \{\xi_{\mathfrak{D}}(\varphi) : (\mathfrak{D}, \varphi) \in \mathfrak{D}\}$  and  $\mathcal{Y} = \{\mathcal{K}_{\mathfrak{D}}(\varphi) : (\mathfrak{D}, \varphi) \in \mathfrak{D}\}$  form a supercharacter theory for  $C_P(\sigma)$ .*

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